

The fractional quantum Hall effect: Chern-Simons mapping, duality, Luttinger liquids and the instanton vacuum

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Abstract

We derive, from first principles, the complete Luttinger liquid theory of abelian quantum Hall edge states. This theory includes the effects of disorder and Coulomb interactions as well as the coupling to external electromagnetic fields. We introduce a theory of spatially separated (individually conserved) edge modes, find an enlarged dual symmetry and obtain a complete classification of quasiparticle operators and tunneling exponents. The chiral anomaly on the edge and Laughlin's gauge argument are used to obtain unambiguously the Hall conductance. In resolving the problem of counter flowing edge modes, we find that the long range Coulomb interactions play a fundamental role. In order to set up a theory for arbitrary filling fractions ν we use the idea of a two dimensional network of percolating edge modes. We derive an effective, single mode Luttinger liquid theory for tunneling processes into the quantum Hall edge which yields a continuous tunneling exponent $1/\nu$. The network approach is also used to re-derive the instanton vacuum or Q -theory for the plateau transitions.

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I. INTRODUCTION

Many of the unusual aspects of both integral and fractional quantum Hall states can be understood from the properties of the excitations at the sample edge. Effective theories for edge excitations take the form of a chiral Fermi liquid in the integral regime or a chiral Luttinger liquid¹ in the fractional regime. These theories are simple examples of so-called conformal field theories that are commonly used as a computational scheme for physical processes such as tunneling into the quantum Hall edge states.²⁻⁵ They also provide an

interesting model system for exploring the thermodynamic consequences of quasiparticle excitations with unusual statistics (exclusion statistics).⁶

The standard Luttinger liquid approach to the quantum Hall edge has remained largely phenomenological and ad hoc in nature. It reflects the idealized properties of "pure" Laughlin states⁷ that are characterized by an *energy gap* and therefore, the analysis is limited to a set of special filling fractions only. But even within its limited range of validity it is not clear how to deal with the fundamental aspects of disorder. For example, controversial issues have arisen regarding the Hall conductance of more complicated states that have edge channels of opposite chirality.^{8,9} Moreover, the lack of a microscopic theory of the fqHe has led to serious discrepancies between experiments on edge tunneling⁵ on the one hand and the Luttinger liquid approach to the quantum Hall edge on the other.

One of the main objectives of the present work is to derive, from first principles, the complete Luttinger liquid theory for abelian quantum Hall states. In our approach to the problem, the physics of the edge appears as an exactly solvable, *strong coupling* limit of a more general theory for arbitrary filling fractions that deals simultaneously with disorder effects and the Coulomb interactions. In a recent series of articles^{10–12}, hereafter referred to as [I], [II] and [III], we have laid the foundation for such a unifying theory. The basic starting point is the Finkelstein sigma model theory (*Q* theory) for localization and interaction effects.¹³ The fundamental principle underlying the unification of compressible and incompressible quantum Hall states is the so-called *instanton vacuum*, a topological concept in quantum field theory that was discovered earlier by one of us in the context of the *free* electron localization theory.¹⁴

In [III] we established the relation between the *instanton vacuum* or *Q*-field approach to the iqHe and the theory of *chiral edge bosons*. In this paper we build on these results and extend the theory to include the fractional effect. For this purpose, we shall employ the idea of flux attachment¹⁵ by coupling the *Q*-field theory to the Chern-Simons statistical gauge fields.^{16,17} Flux attachment transformations have been applied several times before and with different physical objectives.^{18–23} A new ingredient of the present work is that our detailed knowledge of the *integral* quantum Hall edge can now be "mapped" directly onto a complete Luttinger liquid theory of the *fractional* quantum Hall edge. Mathematically, this "mapping" is accomplished by integrating out the Chern-Simons gauge fields over the entire two dimensional plane while confining the electronic degrees of freedom to the half plane. This procedure then leads to the familiar *K*-matrix structure of the edge excitations.^{24–27,1} As a major advancement, however, we shall point out that the flux attachment procedure also provides a simple geometrical interpretation of *duality* transformations²⁸ that invert the *K*-matrix (compactification radius) of the Luttinger liquid edge excitations.

In order to facilitate an analysis of slowly varying potential fluctuations, we introduced in [III] the concept of "spatially separated edge modes". We argued that smooth potential fluctuations near the sample edge decompose the multiple (iqHe) edge states into physically distinguishable and separately conserved edge modes. Since in our theory of the dirty, sharp edge the various edge modes are assumed to scatter strongly amongst each other, it is a priori not entirely obvious whether the concept of spatially separated edge modes "maps" onto a physically equivalent fractionally quantum Hall state under the flux attachment transformation. This problem is closely related to the problem of *counter flowing edge modes* and to a number of puzzles^{29,30} that recently arose in the context of the conventional

K -matrix approach.

By extending the flux attachment transformation to include the case of widely separated edge modes we obtain a rich structure of decomposed fractional quantum Hall edge states which is reminiscent of the picture of multiple condensates in the hierarchy theory of the fqHe.^{31–33} The Luttinger liquid theory for spatially separated edge modes has quite interesting pictorial aspects. In this case we find an *enlarged dual symmetry* of abelian quantum Hall states. We employ the enlarged dual symmetry in order to derive a complete classification of operators for fractionally charged edge excitations as well as tunneling amplitudes.

In the limit of long wavelengths (i.e. for distances large compared to the spatial separation between the edge modes) our theory of separated edge modes reduces in effect to a Luttinger liquid theory of a sharp, clean edge. We next use our explicit results for the "clean" and "dirty" edges to show that they describe identical physics in the limit of large length scales. This not only means that the fractional quantum hall *conductance* is the same in both cases, but also identical amplitudes are obtained for the tunneling of electrons and quasiparticles. For incompressible states with only chiral edge modes these results are precisely what one would naively expect. For more complicated states with both chiral and anti-chiral components present, however, our theory is quite different from the approaches that have been pursued previously.

In view of the fact that the Luttinger liquid theory of the quantum Hall edge has already been extensively investigated following the phenomenological theory of Wen¹, it may be helpful to stress some of the main differences between the standard approach and the development of a microscopic theory of the edge as pursued in this paper.

First, our edge theory is coupled to the external electromagnetic field throughout the derivation. This obviously has considerable technical as well as conceptual advantages. For example, the appearance of the so-called *chiral anomaly*^{28,34} plays a crucial role in relating the Hall conductance as an *edge* phenomenon unambiguously to the Hall conductance as a *bulk* phenomenon. In this way we avoid mistakes in applying ad hoc pictures like the Landauer-Büttiker formalism.⁸ The chiral anomaly also provides an explicit demonstration of Laughlin's gauge argument³⁵ and it plays an important role in identifying the fractionally charged edge excitations in more complex situations.

Secondly, the Chern-Simons mapping or flux attachment procedure is ill-defined in the absence of long range interactions between the particles. For example, without the Coulomb interactions being present, the Hamiltonian for counter flowing edge modes would become unbounded from below. The fundamental significance of long range interactions is necessarily overlooked in the phenomenological approach, where one is free to choose boundary conditions (velocities) and bounded Hamiltonians even for noninteracting particles. The fact that long range Coulomb interactions are needed for a proper CS transformation is related to the central role that they play in the \mathcal{F} -invariance of the Q -field theory.

Thirdly, there is a difference in the way that quasiparticle operators and tunneling exponents are obtained. We shall show that the flux attachment procedure leaves no room for ambiguities, since it directly maps the electron operators of the integral quantum Hall edge into quasiparticle operators of the fractional quantum Hall edge. This procedure discriminates between opposite sides of an edge, thus giving a novel, geometrical interpretation to fractionally charge edge excitations. Physically this means that tunneling processes are described by different particles with different charge and statistics, depending on whether

one tunnels, say, through the vacuum or through the incompressible quantum Hall state.

Finally, our velocity matrix is diagonal simultaneously with K . Since impurities have been dealt with in the first stages of our derivation, they do not make a second appearance in the form of inter-channel scattering of chiral bosons. Consequently, the calculation of quasiparticle and tunneling operators is not plagued by the kind of non-universalities found by^{8,9} for counter flowing charged and neutral modes.

In the second part of this paper we embark on the more general problem of *arbitrary* filling fractions in the fractional quantum Hall regime. We mentioned earlier that our Luttinger liquid theory of edge modes merely describes the strong coupling asymptotics of a unifying theory for arbitrary filling fractions. A complete understanding of this theory is beyond the scope of the present work since it involves a general scaling scenario between extreme physical states that can rarely be followed in a single experiment. If one limits oneself to realistic quantum Hall conditions then a two dimensional, percolating network of edge modes presumably is the appropriate starting point for an analysis of continuously varying filling fractions. Our theory of spatially separated edge modes enables us to formulate such a percolating network, similar to what was introduced previously in the context of the integral quantum Hall regime. In fact, most of the analysis previously done for integral quantum Hall states can be directly "mapped", under the flux attachment transformation, onto the fractional quantum Hall regime. More specifically, the results obtained for the filling fractions varying between m and $m+1$ in the integral regime transform, under an attachment of $2p$ flux quanta per electron, into equivalent statements made on filling fractions varying between ν_m and ν_{m+1} in the fractional regime. Here, $\nu_m = m/(2pm+1)$ denotes the Jain series. Our percolating network of edge modes explains two fundamentally distinct aspects of the fractional quantum Hall regime that cannot be obtained from the theory of Laughlin states or isolated edge excitations alone. These fundamental aspects are easily recognized experimentally. In particular, the *plateau features* that are usually observed on the Hall conductance with varying filling fraction ν are clearly not seen in the recent experiments on electron tunneling⁵ into the quantum Hall edge. The presence or absence of plateau features with varying ν clearly demonstrates the fundamental differences in transport and equilibrium properties of the electron gas.

Following [III] we use the idea of percolating edge states and derive an "effective" Luttinger liquid theory for tunneling processes into the fractional quantum Hall edge. Here, the main idea is that the long range Coulomb interactions between the edge and the localized bulk orbitals give rise to an effective, chiral action of edge excitations in which the neutral modes are strongly suppressed. Only the charged mode contributes to the tunneling processes and the effective Luttinger liquid parameter varies continuously with the filling fraction ν . This yields a tunneling exponent $1/\nu$ in accordance with experiments.

The plateau features in the Hall conductance are explained by the localization of quasiparticles in the bulk and this demands a totally different theory of *plateau transitions*. We shall employ the percolating network model and briefly derive the Q -field theory for localization and interaction effects. The results indicate that the critical aspects of the plateau transitions are quite generally the same for both the integral and fractional regime, independent of the type of disorder (long range vs. short range). However, our results do not exhibit the $Sl(2,Z)$ symmetry that has been proposed and advocated in³⁶⁻³⁸. This symmetry,

therefore, has no microscopic justification and it does not appear as a fundamental (dual) symmetry of the quantum phase transition.

The organization of this paper is as follows. In Section II we describe the details of the flux attachment transformation for the disordered edge. The Q -field or instanton vacuum representation and the equivalent chiral boson theory for the integral quantum Hall regime are reviewed in Section II A. In Section II B we introduce the Chern Simons statistical gauge fields and, by integrating them out, obtain the complete theory for fractional quantum Hall edges. The most important results are summarized by the boson actions of Eqs. (2.17) and (2.22). These include the effects of the Coulomb interaction as well as the coupling of the theory to external electromagnetic fields. Some comments on the problem of counter flowing edge modes are presented in Section II B 4.

In section III we extend the Chern Simons mapping of the disordered edge to include the electron operators and tunneling exponents. We reproduce the Kane-Fisher-Polchinski results⁹ for the tunneling exponents and show the geometrical structure of duality.

In Section IV we introduce the concept of *spatially separated edge channels* and repeat the various steps of flux attachment. A complete set of tunneling exponents is given in (4.5). Section IV B deals with the subject of *enlarged* duality. In Section IV C we compare the Luttinger liquid theory of spatially separated edge modes with theory of the dirty edge. In Section IV D we derive the operators for the Laughlin quasiparticles.

In Section V we apply the idea of percolating edge modes and derive an effective Luttinger liquid theory for tunneling processes away from the special filling fractions.

Section VI is devoted to the Q -theory of the plateau transitions and in Section VII we conclude with a summary of the results.

II. FQHE CHIRAL BOSONS ON A SHARP EDGE

A. Interacting chiral edge bosons for the integer effect

1. Q fields at the edge

In [I-III] we derived the Finkelstein sigma model (Q field) action for disordered, interacting 2D electrons in a strong magnetic field. Specializing to the case where the chemical potential μ lies in an *energy gap* between two Landau bands, the action reduces to a theory for *edge excitations* alone. A detailed understanding of the dynamics of edge excitations is important since it provides invaluable information on the topological concept of an *instanton vacuum* in strong coupling.

In [III] we presented a detailed derivation of the complete "edge" theory coupled to an external electromagnetic field A_μ . Since we are dealing with an exactly soluble, strong coupling limit of an otherwise extremely non-trivial theory we shall proceed by summarizing the results. We refer to [I-III] for a detailed discussion on symmetries, gauge invariance etc. In terms of the matrix field variables $Q_{nm}^{\alpha\beta}$ the edge theory can be expressed as follows

$$S_{\text{eff}}[Q, A] = mS_{\text{top}}[Q] + \frac{im}{4\pi} \left[\sum_{\alpha} \int (A^{\text{eff}})^{\alpha} \wedge d(A^{\text{eff}})^{\alpha} - \beta \sum_{n\alpha} \oint dx (A_x)_{-n}^{\alpha} (A_c^{\text{eff}})_n^{\alpha} \right]$$

$$\begin{aligned}
& + \frac{m\pi}{4\beta v_d} S_F[Q] - \frac{m\pi}{4\beta} \sum_{n\alpha} \int \frac{dk_x}{v^{\text{eff}}(k_x)} \left[\text{tr } I_{-n}^\alpha Q - \frac{\beta}{\pi} (A_c^{\text{eff}})^\alpha_n \right] \left[\text{tr } I_n^\alpha Q - \frac{\beta}{\pi} (A_c^{\text{eff}})^\alpha_{-n} \right] \\
& + \frac{\beta}{2} \left(\frac{m}{2\pi} \right)^2 \sum_{n\alpha} \int d^2x d^2x' B_n^\alpha(\vec{x}) U(\vec{x} - \vec{x}') B_{-n}^\alpha(\vec{x}').
\end{aligned} \tag{2.1}$$

Here, m denotes the integer filling fraction and S_{top} stands for the topological term

$$S_{\text{top}}[Q] = \frac{1}{8} \int d^2x \varepsilon_{ij} \text{tr } Q \partial_i Q \partial_j Q \tag{2.2}$$

which can also be written as an integral over the sample edge. The S_F stands for the one dimensional Finkelstein action

$$S_F[Q] = \oint dx \left[\sum_{n\alpha} \text{tr } I_n^\alpha Q \text{tr } I_{-n}^\alpha Q + 4 \text{tr } \eta Q \right]. \tag{2.3}$$

The explanation of the remaining symbols is as follows. The indices α, β denote replica channels, the n are Matsubara frequency indices and B is the magnetic field. The inverse temperature $(k_B T)^{-1}$ is written as β . The matrix η is given by $\eta_{kl}^{\alpha\beta} = \delta^{\alpha\beta} \delta_{kl} k$.

We have taken the line $y=0$ as the sample's edge and the half-plane $y>0$ as the bulk. The integral $\oint dx$ denotes integration over the edge and $\int d^2x$ should be read as an integral over the upper half-plane. The integral containing the ‘‘wedge’’ notation is defined according to

$$\int A \wedge dA := \int_0^\beta d\tau \int_{-\infty}^\infty dx \int_0^\infty dy \varepsilon^{\alpha\beta\gamma} A_\alpha \partial_\beta A_\gamma, \tag{2.4}$$

where the coordinate x^0 denotes the imaginary time τ . All the fields that we work with in this paper have bosonic (periodic) boundary conditions in the imaginary time coordinate τ . We work in units where $\hbar = e = 1$, and all lengths are expressed in terms of the magnetic length $\ell = \sqrt{2\hbar/eB}$. The superscript $^{\text{eff}}$ indicates that the scalar potential A_0 has acquired a contribution from the Coulomb interactions,

$$A_j^{\text{eff}} = A_j \quad ; \quad A_0^{\text{eff}}(\vec{x}) = A_0(\vec{x}) + \frac{im}{2\pi} \int d^2x' U(\vec{x} - \vec{x}') B(\vec{x}'). \tag{2.5}$$

The subscript ‘ c ’ denotes a chiral direction with an extra interaction term,

$$\begin{aligned}
A_c &= A_0 - iv^{\text{eff}} A_x \quad ; \quad \partial_c = \partial_0 - iv^{\text{eff}} \partial_x \\
v^{\text{eff}}(k_x) &= v_d + \frac{m}{\sqrt{2\pi}} U(k_x) \quad ; \quad v_d = m/(2\pi \rho_{\text{edge}})
\end{aligned} \tag{2.6}$$

where ρ_{edge} is the density of states at the edge. The $U(\vec{x}-\vec{x}')$ is the Coulomb interaction, proportional to $|\vec{x}-\vec{x}'|^{-1}$. The restriction of this interaction to the edge is given (in momentum space) by $U(k_x) = \frac{1}{\sqrt{2\pi}} \int dk_y U(\vec{k})$.

2. Chiral edge bosons

After these preliminaries we quote a completely equivalent but much simpler theory for edge dynamics which is expressed in terms of chiral edge boson fields ϕ_i with the index i running from 1 to $|m|$. In [III] we obtained the complete chiral boson action as follows

$$\begin{aligned}
S_{\text{chiral}} = & \frac{i}{4\pi} \text{sgn}(m) \sum_{i=1}^{|m|} \left[\int A \wedge dA - \oint (D_x \varphi_i D_- \varphi_i - E_x \varphi_i) \right] \\
& - \frac{1}{2} \left(\frac{1}{2\pi} \right)^2 \sum_{i,j=1}^{|m|} \int_{\infty} d\tau d^2 x d^2 x' U(\vec{x} - \vec{x}') \nabla \times [\theta(y) \vec{D} \varphi_i(\vec{x})] \nabla' \times [\theta(y') \vec{D} \varphi_j(\vec{x}')].
\end{aligned} \tag{2.7}$$

Here the covariant derivative is defined as $D_\mu \varphi = \partial_\mu \varphi - A_\mu$. The θ is the Heaviside step function and $D_- \varphi = D_0 \varphi - i v_d D_x \varphi$. We use the shorthand notation \oint for $\int_0^\beta d\tau \oint dx$. Notice that there are no ‘effective’ quantities in (2.7); The Coulomb interaction is completely contained in the last term. The charge density is given by $\frac{m}{2\pi} [B + \delta(y) |m|^{-1} \sum_i D_x \varphi_i]$. Notice also that we have written a two-dimensional integral containing φ_i , even though the boson fields only exist on the edge. This is not a problem, since the φ_i only get evaluated at the edge.

The equivalent theories of edge dynamics, (2.1) and (2.7), provide an important tool for investigating the problem of “long range potential fluctuations” in quantum Hall systems. For example, in [III] we introduced a 2D network of percolating edge modes as a model for long range potentials and arbitrary filling fractions ν . It was shown that tunneling processes at the quantum Hall edge are conveniently described by an *effective* theory of chiral edge bosons. The physics of the plateau transitions, on the other hand, is completely different and the original instanton vacuum or Q theory can be re-derived from the network model, starting from (2.1).

The Chern Simons mapping, described below, proceeds entirely in terms of the chiral boson version (2.7). We return to the Q fields in Section VI where we embark on the subject of the plateau transitions.

B. The fractional effect

1. Composite fermions

In writing (2.7), we have included the possibility of m being negative. The reason for this is that we want to be able to perform Jain’s composite fermion mapping¹⁵, which relates integer filling m to fractional values ν , in such a way that the electron and the composite fermion experience magnetic fields of opposite sign. We use a convention in which ν is always positive and the magnetic field of the filling fraction ν sample points in the positive z direction. Since particle densities are positive, we then have to allow negative m .

We implement the composite fermion picture in the standard way by introducing statistical gauge fields a_μ . The action for the a_μ has the form of a Chern-Simons action^{16,17} and our starting point can be written as

$$S = S_{\text{chiral}}[\varphi_i, A + a] + \frac{i}{4\pi} \cdot \frac{1}{2p} \int_{\infty} a \wedge da \tag{2.8}$$

with S_{chiral} given by (2.7). Here p is a positive integer and the subscript ∞ denotes that the spatial integral is over the whole xy -plane instead of just the upper half.

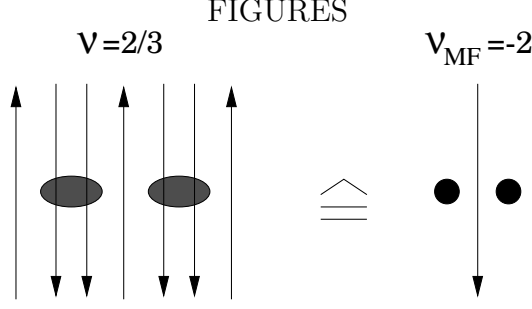


FIG. 1. Equivalence between filling fraction $2/3$ of electrons coupled to a CS gauge field (left) and filling -2 in mean field theory (right). Arrows denote flux quanta. The dots represent electrons.

Several remarks should be added to this result. First of all, the theory of (2.8) does not contain the zero frequency modes and it really stands for the *fluctuations* about the Chern Simons mean field theory. As a result, the S_{chiral} in (2.8) is defined for an effective magnetic field B^{eff} resulting from a “smearing out” of the statistical fluxes. It is given by [I]

$$B^{\text{eff}} = B - 2pn_e, \quad (2.9)$$

with n_e the electron density. The $2p$, on a mean field level, stands for the number of elementary flux quanta attached to each electron. The only effect of (2.8) is that the integer filling fraction m now refers to *composite fermions* rather than electrons. The true filling fraction ν of the electron gas is related to m in the following way,

$$\nu = \frac{m}{2pm + 1}, \quad (2.10)$$

where m can be negative if the signs of B and B^{eff} are not the same. As an illustration we show in Fig. 1 that the $\nu=2/3$ state is equivalent to integer filling $m=-2$ at a mean field level if two flux quanta per electron are applied.

Secondly, it is important to stress that the statistical gauge fields a_μ are defined over the entire plane. The electronic part S_{chiral} is restricted to the half plane and this, properly regarded, is the result of a confining potential added to the electron gas.

Keeping these preliminary remarks in mind we next turn to the problem of the Chern Simons fluctuating gauge fields.

2. Integrating out the CS fluctuations

We note that the action can be written in such a way that $a_- (= a_0 - iv_d a_x)$, the component in the “preferred” chiral direction, evidently multiplies a constraint,

$$\begin{aligned} S = & \frac{i}{4\pi} \left[\frac{1}{2p} \int_{\infty} (2a_- \nabla \times \vec{a} - \vec{a} \times \partial_- \vec{a}) \right. \\ & + \text{sgn}(m) \sum_{i=1}^{|m|} \int_{\infty} \left(2\mathcal{D}_- \varphi_i \nabla \times [\theta(y) \vec{\mathcal{D}} \varphi_i] - \theta(y) \vec{\mathcal{D}} \varphi_i \times \partial_- \vec{\mathcal{D}} \varphi_i \right) \\ & \left. - \frac{1}{2} \left(\frac{1}{2\pi} \right)^2 \sum_{i,j=1}^{|m|} \int_{\infty} U(\vec{x} - \vec{x}') \nabla \times [\theta(y) \vec{\mathcal{D}} \varphi_i(\vec{x})] \nabla' \times [\theta(y') \vec{\mathcal{D}} \varphi_j(\vec{x}')]. \right] \end{aligned} \quad (2.11)$$

Here \mathcal{D} is defined as a covariant derivative containing both gauge fields, $\mathcal{D}_\mu \varphi_i = \partial_\mu \varphi_i - A_\mu - a_\mu$. The remarkable thing about (2.11) is that the action for the edge bosons is completely contained in *bulk* integrals. Derivatives of the step functions with respect to the y coordinate lead to the edge terms in (2.7). The constraint multiplied by a_- is given by

$$0 = \nabla \times \left[\vec{a} - 2p \operatorname{sgn}(m) \theta(y) \sum_i \vec{\mathcal{D}} \varphi_i \right] \quad (2.12)$$

with general solution

$$\vec{a} = \frac{2p}{2pm\theta(y) + 1} \operatorname{sgn}(m) \left[\theta(y) \sum_i \vec{\mathcal{D}} \varphi_i + \nabla \Omega \right] \quad (2.13)$$

where Ω is an arbitrary function. Substitution of (2.13) into what remains of (2.11) after integration of a_- yields an expression containing only the following linear combinations

$$\tilde{\varphi}_i := \varphi_i - 2p \operatorname{sgn}(m) \Omega. \quad (2.14)$$

The mapped action has a form similar to (2.11),

$$\begin{aligned} S = & \frac{i}{4\pi} \sum_{ij} \left[\delta_{ij} \operatorname{sgn}(m) - 2p \frac{\nu}{m} \right] \int_\infty \left(2D_- \tilde{\varphi}_i \nabla \times [\theta(y) \vec{D} \tilde{\varphi}_j] - \theta(y) \vec{D} \tilde{\varphi}_i \times \partial_- \vec{D} \tilde{\varphi}_j \right) \\ & - \frac{1}{2} \left(\frac{\nu}{2\pi m} \right)^2 \sum_{ij} \int_\infty U(\vec{x} - \vec{x}') \nabla \times [\theta(y) \vec{D} \tilde{\varphi}_i(\vec{x})] \nabla' \times [\theta(y') \vec{D} \tilde{\varphi}_j(\vec{x}')]. \end{aligned} \quad (2.15)$$

See appendix A for the explicit calculation. In the expression $[\dots]$ we recognize the inverse of the coupling matrix (“ K -matrix”) that was obtained in Chern-Simons theories,^{24–27,39}

$$K_{ij} = \delta_{ij} \operatorname{sgn}(m) + 2p. \quad (2.16)$$

Bringing the action (2.15) into the form (2.7) we obtain

$$\begin{aligned} S = & \frac{i\nu}{4\pi} \int A \wedge dA - \frac{i}{4\pi} \sum_{ij} (K^{-1})_{ij} \oint (D_x \tilde{\varphi}_i D_- \tilde{\varphi}_j - \tilde{\varphi}_i E_x) \\ & - \frac{1}{2} \left(\frac{\nu}{2\pi m} \right)^2 \sum_{ij} \int_\infty U(\vec{x} - \vec{x}') \nabla \times [\theta(y) \vec{D} \tilde{\varphi}_i(\vec{x})] \nabla' \times [\theta(y') \vec{D} \tilde{\varphi}_j(\vec{x}')]. \end{aligned} \quad (2.17)$$

Eq. (2.17) is the main result of this chapter. Even though bits and pieces of this action, such as the coupling of $\tilde{\varphi}$ to the gauge field²⁸, have appeared in the literature before and the K -matrix structure is wholly familiar, our complete theory is new, as many aspects of the impurity problem were previously not understood.

The Hall conductance on the edge can unambiguously be determined from (2.17) by using the chiral anomaly.^{28,34} The chiral anomaly enables us to use Laughlin’s flux argument on the edge, since it relates the divergence of the edge current to the electric field parallel to the edge,

$$\partial_\mu J_{\text{edge}}^\mu = -\frac{1}{2\pi} \sigma_H E_x \quad ; \quad J_{\text{edge}}^\mu = \frac{\delta S}{\delta A_\mu} \Big|_{\text{edge}}. \quad (2.18)$$

In this way we easily obtain that the Hall conductance on the edge σ_H is equal to ν , as it should be.

In order to make contact with phenomenological models¹, we mention that we can write down a bulk Chern-Simons action that is completely equivalent to (2.17). The electronic degrees of freedom are contained in fields g_i that act as potentials for the current,

$$S = \frac{i}{4\pi} \sum_{ij=1}^{|m|} (K^{-1})_{ij} \left[- \int g^i \wedge dg^j + 2 \int g^i \wedge dA^{\text{eff}} \right] + \frac{\nu^2}{8\pi^2} \int U(\vec{x} - \vec{x}') B(\vec{x}) B(\vec{x}'). \quad (2.19)$$

The g^i satisfy the following gauge fixing constraint at the edge

$$\left[g_-^i(k_x) - i \frac{m}{\sqrt{2\pi}} U(k_x) \sum_{a=1}^{|m|} g_x^a(k_x) \right]_{\text{edge}} = 0. \quad (2.20)$$

The free part of (2.19) is known from the work of Wen¹. However, lacking a microscopic edge theory, the full details of the interactions were unknown.

3. Charged and neutral modes

Diagonalization of the K -matrix yields one charged mode and $|m| - 1$ neutral modes. The charged mode Γ is given by

$$\Gamma = \frac{1}{|m|} \sum_i \tilde{\varphi}_i \quad (2.21)$$

where the normalization has been chosen such that Γ has unit charge. A possible choice of basis for the neutral modes g_a is

$$g_a = \frac{1}{a} \sum_{k=1}^a \tilde{\varphi}_k - \tilde{\varphi}_{a+1} \quad a = 1, \dots, |m| - 1 \quad (2.22)$$

$$\tilde{\varphi}_k = \Gamma - \frac{k-1}{k} g_{k-1} + \sum_{a=k}^{|m|-1} \frac{1}{a+1} g_a.$$

The Jacobian of the transformation from the $\tilde{\varphi}_i$ to the modes (2.21, 2.22) is unity. In diagonalized form the action reads

$$S = \frac{i\nu}{4\pi} \left[\int A \wedge dA - \oint (D_x \Gamma D_- \Gamma - \Gamma E_x) \right] - \frac{i}{4\pi} \text{sgn}(m) \sum_{a=1}^{|m|-1} \frac{a}{a+1} \oint \partial_x g_a \partial_- g_a$$

$$- \frac{1}{2} \left(\frac{\nu}{2\pi} \right)^2 \int_{\infty} U(\vec{x} - \vec{x}') \nabla \times [\theta(y) \vec{D} \Gamma(\vec{x})] \nabla' \times [\theta(y') \vec{D} \Gamma(\vec{x}')]. \quad (2.23)$$

Naturally, the charged mode is the only one coupling to A_μ and feeling interactions.

4. (In)stability of counter-flowing channels

Here we make an important observation which has been missed in the literature, namely that *for the noninteracting case or for short-range interactions the Chern-Simons mapping is ill-defined at negative m* . In order to see this, we look at the Hamiltonian for the charged mode corresponding to (2.23) in the absence of Coulomb interactions,

$$H_{\Gamma}^{(\text{free})} = \frac{v_d \nu}{4\pi} \oint dx (D_x \Gamma)^2. \quad (2.24)$$

Recall that v_d is defined as $m/(2\pi\rho_{\text{edge}})$ and that $\nu > 0$. Hence, for negative m the free Hamiltonian is unbounded from below! The Coulomb interactions cure this situation by adding an extra term to the edge Hamiltonian,

$$H_c = \frac{1}{2} \left(\frac{\nu}{2\pi} \right)^2 \oint dx dx' U(x, x') D_x \Gamma(x) D_x \Gamma(x') \quad (2.25)$$

effectively changing the velocity v_d to

$$\tilde{v}(k_x) = v_d + \frac{\nu}{\sqrt{2\pi}} U(k_x) \quad , \quad U(k_x) \propto \ln(k_x) \quad (2.26)$$

For low momenta the logarithm in $\tilde{v}(k_x)$ is dominant, ensuring that the effective velocity \tilde{v} is positive even if $v_d < 0$ and thereby yielding a bounded Hamiltonian. We conclude that the long range Coulomb interactions are actually needed in order for the Chern-Simons mapping to be well defined. A similar situation occurs in the bulk theory with half integer filling fraction [I] where in the absence of long range interactions the CS procedure gives rise to a divergent quasiparticle d.o.s.

III. TUNNELING DENSITY OF STATES

A. The integer case

In [III] we obtained the following results for the IQHE. In the Q -field theory, the tunneling d.o.s. for tunneling electrons from a Fermi liquid into the edge of a quantum Hall system is given by

$$\langle Q^{\alpha\alpha}(\tau_2, \tau_1, \vec{x}_0) \rangle := \sum_{n=-\infty}^{\infty} e^{i\nu_n(\tau_2 - \tau_1)} \langle Q_{nn}^{\alpha\alpha}(\vec{x}_0) \rangle \quad (3.1)$$

where \vec{x}_0 is the position where the tunneling electron enters the sample, τ_1 and τ_2 are two instants in imaginary time and ν_n is the bosonic Matsubara frequency $\frac{2\pi}{\beta}n$. The process of integrating out Q from the action (2.1) changes (3.1) to

$$\left\langle \exp -i \left[\partial_c^{-1} A_c^{\text{eff}}(\tau_2, \vec{x}_0) - \partial_c^{-1} A_c^{\text{eff}}(\tau_1, \vec{x}_0) \right] \right\rangle. \quad (3.2)$$

The process of decoupling the quadratic edge term in A_μ by the introducing of boson fields φ_i transforms (3.2) into

$$\left\langle \exp -i \int_{\tau_1}^{\tau_2} d\tau \partial_0 \varphi_j(\tau, \vec{x}_0) \right\rangle, \quad j = 1, \dots, |m| \quad (3.3)$$

where the expectation value is now taken with respect to the φ action (2.7). Any of the fields $\varphi_1, \dots, \varphi_{|m|}$ can be put in the exponent, since they are interchangeable.

The completely equivalent edge theories in terms of Q (2.1) and in terms of φ_i (2.7) now ensure that the expressions (3.1) and (3.3) are identical. An explicit computation leads to the same Fermi liquid result

$$(\tau_2 - \tau_1)^{-S} \quad ; \quad S = 1. \quad (3.4)$$

in both cases.

B. The fractional case

1. Results of the Chern-Simons mapping

We next wish to extend the CS mapping procedure to include the result for the electron propagator at the edge (3.3). This problem is nontrivial since the amount of statistical flux that is trapped now depends on whether the path from time τ_1 to τ_2 is taken just outside or just inside the sample. Hence, the obvious translation of (3.3) into composite fermion language,

$$\left\langle \exp -i \int_{\tau_1}^{\tau_2} d\tau [\partial_0 \varphi_j - a_0](\tau, \vec{x}_0) \right\rangle, \quad (3.5)$$

now has an additional *geometrical* significance. In order to find out what this geometrical significance is, we must repeat the steps (2.11–2.17), but now in the presence of (3.5).

We write $a_0 = a_- + i\nu a_x$. The presence of a_- at \vec{x}_0 modifies the constraint equation (2.12) and we get

$$\nabla \times \left[\vec{a} + 2p \operatorname{sgn}(m) \theta(y) \sum_i (\vec{a} - \vec{D} \varphi_i) \right] = -2p \cdot 2\pi \delta(\vec{x} - \vec{x}_0) L \quad (3.6)$$

where we have defined a quantity L such that

$$\beta \sum_n L_{-n} f_n(\vec{x}) = \int_{\tau_1}^{\tau_2} d\tau f(\tau, \vec{x}) \quad (3.7)$$

for any bosonic function f . The factor of 2π in (3.6) originates from our convention $\hbar = 1$, while a flux quantum is given by h/e , and the minus sign shows that the statistical flux indeed points in the negative z direction. From (3.6) we see that the amount of flux is given by

$$2p \text{ for } y_0 < 0 \quad ; \quad 2p\nu/m \text{ for } y_0 > 0. \quad (3.8)$$

This is an important result. It shows that for $y_0 > 0$ the CS mapping has altered the charge of the tunneling particle from 1 into ν/m . A physical interpretation is readily given in terms of

Laughlin's gauge argument. If one flux quantum is sent through a hole in the IQH sample, one electron will leave the edge per filled Landau level, resulting in a total charge m . The CS mapping changes the transported charge, which is equal to the Hall conductance, to ν . Since this charge is contained in m Landau levels, the charge per Landau level has to be ν/m .

We will now integrate out the Chern-Simons gauge field. The general solution to the constraint equation (3.6) is

$$\vec{a}(\vec{x}) = \frac{2p \operatorname{sgn}(m)}{2pm\theta(y)+1} \left[\theta(y) \sum_i \vec{D}\varphi_i + \nabla\Omega + \operatorname{sgn}(m)L\nabla\operatorname{arctg}\frac{x-x_0}{y-y_0} \right] \quad (3.9)$$

with Ω again an arbitrary gauge. Substituting this solution \vec{a} into what remains of the action, we re-obtain the action (2.17), but now with a slightly different definition of the $\tilde{\varphi}$ fields,

$$\tilde{\varphi}_i = \varphi_i - 2p \operatorname{sgn}(m)\Omega - 2pL \operatorname{arctg}\frac{x-x_0}{y-y_0}. \quad (3.10)$$

See appendix B for the details of this calculation. The expectation value acquires the following form after the CS mapping

$$\exp -i \int_{\tau_1}^{\tau_2} d\tau \left\{ \partial_0 \tilde{\varphi}_i - \theta(y_0) 2p \frac{\nu}{|m|} \sum_j D_0 \tilde{\varphi}_j \right\} (\tau, \vec{x}_0). \quad (3.11)$$

For further considerations we put $A_\mu=0$, since the tunneling is done at zero gauge field.

Several remarks need to be made here. First of all, we see that the outcome will indeed not depend on the gauge Ω . Secondly, the expectation value will be different for \vec{x}_0 going to the edge from inside or from outside of the sample. Furthermore, there is a difference between the cases $m>0$ and $m<0$.

If we take $y_0 \uparrow 0$, we easily obtain the result (see appendix C)

$$\left\langle \exp -i \int_{\tau_1}^{\tau_2} d\tau \partial_0 \tilde{\varphi}_i \right\rangle = (\tau_2 - \tau_1)^{-S_{\text{out}}} \quad ; \quad S_{\text{out}} = 2p + 1 + \left(\frac{1}{m} - \frac{1}{|m|} \right) \quad (3.12)$$

which is the tunneling exponent found by⁹.

For positive m we have $S_{\text{out}} = K_{ii} = \text{odd}$, which clearly indicates a composite fermion. For negative m the meaning of the exponent is not clear; it does not in general represent a fermionic quantity.

For $y_0 \downarrow 0$ we can rewrite (3.11) as

$$\exp \left[-i \operatorname{sgn}(m) \int_{\tau_1}^{\tau_2} d\tau \partial_0 \sum_j K_{ij}^{-1} \tilde{\varphi}_j(\tau, \vec{x}_0) \right] \quad (3.13)$$

Expression (3.13) gives us a tunneled charge equal to $\operatorname{sgn}(m) \sum_j K_{ij}^{-1} = \nu/m$ in accordance with (3.8). Note that the charge is not contained in one Landau channel, as was the case in the tunneling expression for the integer effect, but instead spread out over all the channels. In the i 'th channel there is a charge $1 - 2p\nu/|m|$, in all the other channels $-2p\nu/|m|$.

It is readily seen that the tunneling exponent becomes

$$S_{\text{in}} = 1 - 2p \frac{\nu}{m} + \left(\frac{1}{m} - \frac{1}{|m|} \right). \quad (3.14)$$

For positive m this expression takes the form $S_{\text{in}} = K_{ii}^{-1} = 1 - 2p\nu/m$.

2. Duality

Notice that the cases $y_0 \uparrow 0$ and $y_0 \downarrow 0$ are related by a “T-duality” in the sense of ref.²⁸, with K playing the role of compactification radius. If we define “dual” fields ξ_i with charge ν/m as $\xi_i = \text{sgn}(m) K_{ij}^{-1} \tilde{\varphi}_j$, then the action (2.17) takes the form

$$S[\xi] = \frac{i\nu}{4\pi} \int A \wedge dA - \frac{i}{4\pi} \sum_{ij} K_{ij} \oint \left(D_x \xi_i D_{-x} \xi_j - \frac{\nu}{m} \xi_i E_x \right) - \frac{1}{2} \left(\frac{1}{2\pi} \right)^2 \sum_{ij} \int_{\infty} U(\vec{x} - \vec{x}') \nabla \times [\theta(y) \vec{D} \xi_i(\vec{x})] \nabla' \times [\theta(y') \vec{D} \xi_j(\vec{x}')], \quad (3.15)$$

with $D_\mu \xi_i = \partial_\mu \xi_i - \frac{\nu}{m} A_\mu$. The exponent (3.13) is written as

$$\exp[-i \int_{\tau_1}^{\tau_2} d\tau \partial_0 \xi_i], \quad (3.16)$$

i.e. the same form as (3.12) but now in terms of ξ . The action (3.15) for the ξ fields contains the matrix K instead of K^{-1} . In²⁸ it was shown that the energy spectrum for a system of 1+1D coupled bosons is invariant under $O(m, m, Z)$ transformations acting on a specific $2m \times 2m$ matrix that expresses the Hamiltonian in terms of the momenta and winding numbers of a state. The replacement $K \rightarrow K^{-1}$ is an example of such an $O(m, m, Z)$ transformation. In a certain way it interchanges fluxes and particles.

In the above considerations the statistics parameter is calculated of a particle with charge one; the amount of statistical flux that it acquires is either $2p$ (outside the sample) or $2p\nu/m$ (inside). If we were, on the other hand, to concentrate on the flux instead of the charge, we would consider the case of $2p\nu/m$ statistical flux quanta which acquire *charge*, namely 1 inside the sample and ν/m outside. In this sense there is charge-flux duality.

Notice that the roles of inside/outside get reversed in the argument given above. We make this more explicit by defining

$$p' = -p \frac{\nu}{m}. \quad (3.17)$$

The duality relation $K \rightarrow K^{-1}$ can be written as

$$K(p', m) = K^{-1}(p, m). \quad (3.18)$$

In Fig. 2 it is shown that an exchange of p to p' interchanges the opposite sides of the edge. We will come back to this point in section IV B.

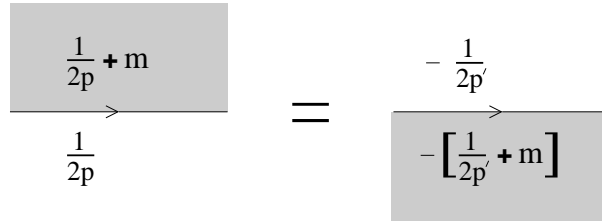


FIG. 2. Inverse number of statistical flux quanta inside and outside the sample. The shaded area denotes the inside. The duality $p \leftrightarrow p'$ reverses the role of the opposite sides of the edge.

In the case of the simple Laughlin fractions $\nu = 1/(2p+1)$ the duality is particularly transparent. The fact that there is only one edge channel means that there is no redistribution of charge over various channels, which allows for an easy interpretation of the tunneling exponent in terms of flux-charge composites. Let us consider the complex conjugate of (3.13); taking the complex conjugate does not change the tunneling exponent, but it changes a particle to a hole. Expression (3.13) becomes $\exp -i \int d\tau (-\nu) \partial_0 \tilde{\varphi}$, describing the tunneling of a hole of charge $-\nu$. The tunneling exponent is given by $S_{\text{in}} = \nu$. We can interpret this result as follows. The hole has a statistical flux $-2p\nu$ attached to it. In addition, it has an “intrinsic” flux of $-\nu$ coming from the fermionic statistics of the electron. The statistical parameter of a charge-flux composite is given by the product of charge and flux, in this case $(-\nu) \cdot (-2p\nu - \nu) = \nu$ for the quasi-hole. The relation $K \rightarrow K^{-1}$ takes the simple form $\nu^{-1} \rightarrow \nu$.

IV. LONG RANGE DISORDER

A. Separation of edge channels; Edge particles

In [III] we discussed how long range disorder can cause the edge states of different Landau levels to become spatially separated. A potential fluctuation at the edge generically lifts all states in such a way that new ‘edges’ are created. If the chemical potential lies between the shifted and unshifted energy of a Landau level, the edge states of this Landau level will be situated inside the sample, not on the outermost edge. If there are several potential jumps of this kind, all the edge channels can become separated. Viewed from above, the edge channels will mostly lie closely together at the edge, while at certain points one or more of them go venturing into the bulk.

This picture leads us to the idea of describing the chiral bosons by *one* field $\varphi(\vec{x})$ that lives on $|m|$ edges instead of $|m|$ fields that live on one edge. The action for the integer effect (2.7) then becomes

$$S = \frac{i}{4\pi} \left[\int n A^\alpha \wedge dA^\alpha - \text{sgn}(m) \sum_{a=1}^{|m|} \oint_a (D_x \varphi D_- \varphi - E_x \varphi) \right] - \frac{1/2}{(2\pi)^2} \int_{x,x'} U(\vec{x} - \vec{x}') \nabla \times [n(\vec{x}) \vec{D} \varphi(\vec{x})] \nabla' \times [n(\vec{x}') \vec{D} \varphi(\vec{x}')] \quad (4.1)$$

where $n(\vec{x})$ stands for the local filling: outside the sample $n(\vec{x})$ is zero; every time you cross an edge it changes its value by one, until it reaches its bulk value m . The \oint_a denotes integration over the a 'th edge. Every edge is given a label $a \in Z$, equal to the maximum filling that the edge is bordering on. For positive m we have $a \in \{1, \dots, m\}$, for negative m $a \in \{m+1, \dots, 0\}$.

So we now have regions in which the integer effect occurs at different values, separated by edges of the $\nu = 1$ type on which the boson field φ lives. Performing the Chern-Simons mapping to the fractional effect simply maps the filling fractions in these regions separately from $n(\vec{x})$ to $\nu(\vec{x})$,

$$\nu(\vec{x}) = \frac{n(\vec{x})}{2pn(\vec{x})+1}. \quad (4.2)$$

The mapped action reads

$$S = \frac{i}{4\pi} \left[\int \nu A \wedge dA - \sum_{a=1}^{|m|} \Delta \nu_{[a]} \oint_a (D_x \Phi D_- \Phi - E_x \Phi) \right] - \frac{1/2}{(2\pi)^2} \int_{x,x'} U(\vec{x} - \vec{x}') \nabla \times [\nu(\vec{x}) \vec{D} \Phi(\vec{x})] \nabla' \times [\nu(\vec{x}') \vec{D} \Phi(\vec{x}')]. \quad (4.3)$$

[A redefinition of the type (3.10) has occurred and the new field is denoted as Φ .] Integration over the a 'th edge counting from the vacuum is written as \oint_a . We use the label a both for edges and for regions: the region lying “above” the a 'th edge also carries the label a . The $\Delta \nu_{[a]}$ is defined as $\nu_a - \nu_{a-1}$ for positive m and $\nu_{-a} - \nu_{-(a-1)}$ for negative m ; here ν_a stands for the local filling fraction in region number a . In Fig. 3 we have made an overview (for samples with both signs of m in the same figure) of the parameters a , ν_a and $\Delta \nu_a$ in the simple case $p = 1$.

The tunneling term is simply a generalization of (3.11),

$$\left\langle \exp -i \int_{\tau_1}^{\tau_2} d\tau [1 - 2p\nu(\vec{x}_0)] \partial_0 \Phi(\tau, \vec{x}_0) \right\rangle, \quad (4.4)$$

where the expectation value is taken with respect to (4.3). The result again depends on the direction from which the edge is approached. The tunneled charge is given by $q_a = \nu_a/a$ if an edge is approached from region number a .

Evaluating (4.4) is a nontrivial exercise, since the Coulomb interaction couples the edges to each other.

If we imagine the edges to be far apart, only the intra-edge interaction is still present, drastically simplifying the calculation. Tunneling exponents are then given by the simple expression $S = q^2/\Delta \nu$. Labeling an approach from above by ‘H’ and from below by ‘L’, we write the tunneling exponents at the a 'th edge ($a \neq 0$) as follows

$$S_a^L = 1 + 2p \frac{\nu_{a-1}}{a-1} = \frac{a}{\nu_a} \cdot \frac{\nu_{a-1}}{a-1} \\ S_a^H = 1 - 2p \frac{\nu_a}{a} = \frac{\nu_a}{a} \cdot \frac{a-1}{\nu_{a-1}} \quad ; \quad S_a^L S_a^H = 1. \quad (4.5)$$

The edge $a=0$ represents the outer boundary of a negative- m sample. Here we get

$$S_0^H = S_0^L = 2p - 1. \quad (4.6)$$

(See Fig. 3.) Notice that in this approach, with spatially separated edge channels, the vacuum exponent for $m < 0$ is always fermionic. Notice also that the channel labeled $a=0$, lying between the vacuum and an $m < 0$ sample, is the only one that needs long-range Coulomb interactions in order to be stable. In Fig. 3 this channel has a chirality opposite to all the others.

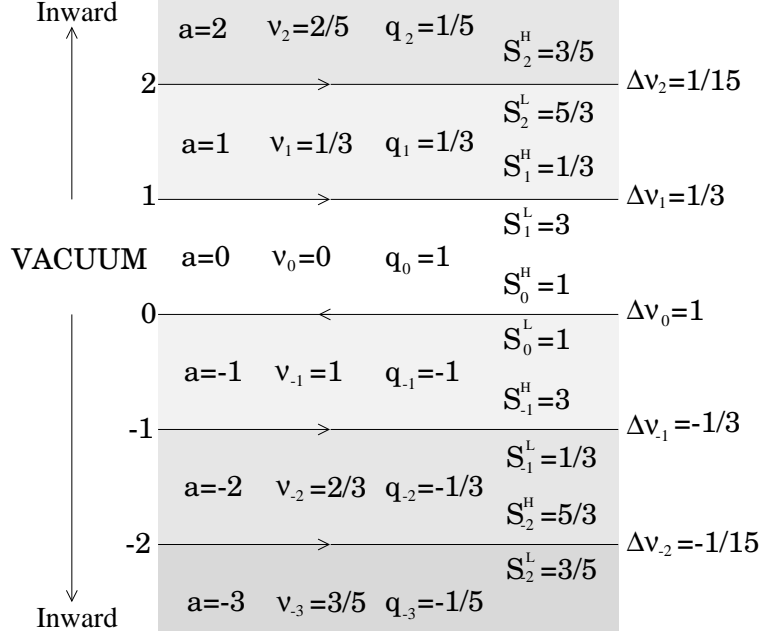


FIG. 3. Edge quasiparticle duality for $2p = 2$. A schematic drawing is given of the regions with different $\nu(\vec{x})$; systems with $m < 0$ and $m > 0$ are combined in one diagram. Arrows represent the chirality of an edge. Given are the local filling fraction, the charge of the tunneling particle as well as the tunneling exponent S on both sides of each edge.

B. Enlarged duality

As mentioned in section IIIB, the Hamiltonian of the chiral boson model has a duality symmetry under $K \rightarrow K^{-1}$. In the case of spatially separated edge channels, this duality transformation has an extra geometrical interpretation: *The order of the edge channels gets reversed*. The transformation $K \rightarrow K^{-1}$ can be accomplished by transforming $p \rightarrow p'$, with

$$2p' = -2p\nu/m. \quad (4.7)$$

The easiest way to visualize the reversal of channels is to consider how the transformation (4.7) affects the amount of statistical flux that a tunneling particle carries (see Fig. 4). The inverse number of flux quanta is $1/2p$ outside the sample and increases with unit steps every time an edge is crossed, until the value $m+1/2p$ is reached in the bulk proper. Writing (4.7) as $-\frac{1}{2p'} = \frac{1}{2p} + m$, we see that

$$-(\frac{1}{2p'} + a) = \frac{1}{2p} + (m - a). \quad (4.8)$$

The minus sign preserves the chirality of the edge particles.

More generally, under a transformation

$$p \rightarrow p' = -p\nu_b/b, \quad b \in Z \quad (4.9)$$

the whole structure of Fig. 3, including region labels, edge labels, filling fractions and tunneling exponents, gets mirrored in the line $a = a_{\text{mirror}}$,

$$a_{\text{mirror}} = \frac{b+1}{2}. \quad (4.10)$$

This results in the following relations between tunneling exponents at different edges:

$$\begin{aligned} S_a^{\text{H}}(p') &= S_{b+1-a}^{\text{L}}(p) \quad ; \quad S_a^{\text{L}}(p') = S_{b+1-a}^{\text{H}}(p) \\ a \neq 0 \quad ; \quad b+1-a &\neq 0. \end{aligned} \quad (4.11)$$

Unless $b = -1$, expression (4.11) does not work for the mirrored tunneling exponent of the counter-flowing edge $a = 0$ and its mirror image $a = b+1$.

$$S_{b+1}^{\text{H}}(p') = S_{b+1}^{\text{L}}(p') = 1 - 2p = -S_0(p) \quad (4.12)$$

$$S_0(p') = -(1 + 2p\nu_b/b) = \begin{cases} -S_{b+1}^{\text{L}}(p) & \text{for } b \neq -1 \\ S_0(p) & \text{for } b = -1 \end{cases} \quad (4.13)$$

There are two special cases:

- $b = -1$, where the symmetry axis lies exactly on the counter-flowing channel. Every region gets mapped into minus itself, while the flux is left unchanged! ($p' = p$.) This shows that at given p , all tunneling exponents are symmetric under ($a \rightarrow -a$, $\text{L} \leftrightarrow \text{H}$). The vacuum gets mapped into the $a = -1$ region, which is suggestive of an interpretation of this region as a filled $\nu = 1/(2p-1)$ ‘vacuum’ from which holes can be created.
- $b = 0$, with the symmetry axis at $a = 1/2$. We have $p' = -p$, so we are in a situation with reversed magnetic field. This transformation relates samples with composite fermion filling m to those with $-m$.

$$\begin{array}{ccc} \frac{1}{2p} + m & & -\frac{1}{2p'} \\ \hline \rightarrow & & \rightarrow \\ \vdots & & -\left[\frac{1}{2p'} + 1\right] \\ \vdots & & \hline \rightarrow & = & \rightarrow \\ \frac{1}{2p} + 1 & & \vdots \\ \hline \rightarrow & & \rightarrow \\ \frac{1}{2p} & & -\left[\frac{1}{2p'} + m\right] \end{array}$$

FIG. 4. Geometrical interpretation of the duality relation $K \rightarrow K^{-1}$ as a reversal of edge channels. For each region the inverse number of statistical flux quanta is given. ($m > 0$)

C. Relation between dirty and clean edges

We considered the various edge modes to be far apart. The results for the exponents (4.5) are valid provided that one probes the system at length scales short relative to the distance between the edge modes. In this section, we consider the action (4.3) in the opposite limit of large distances. The edge modes then strongly interact and (4.5) now becomes a Luttinger liquid theory of a sharp, “clean” edge. It is not a priori obvious that this theory is physically the same as the theory of the sharp “dirty” edge. In the following we show that this is indeed the case. In particular, the two theories yield identical results for the two kinds of tunneling exponent that can be found from (3.11) in the “dirty” $\tilde{\varphi}$ -theory, namely the case where the limit to the edge is taken from the vacuum and the case where one approaches the edge from inside the sample.

If one writes Φ_i for the field Φ evaluated on the i 'th edge then it is possible to express the fields Φ_i in terms of a charged mode Γ and neutral modes $\gamma_1, \dots, \gamma_{|m|-1}$ in such a way that the action (4.3) for the Φ_i exactly matches the action (2.17) for the $\tilde{\varphi}_i$. (In principle, the charged and neutral modes are not locally defined fields, since they contain parts from different edges. However, in the long distance limit all the edges are compressed together on one contour and the fields are local.) The relation between the Φ_i and the $\tilde{\varphi}_i$ is nontrivial; we show that the fields are dual to each other for $m > 0$ and explain why this leads to identical numbers when one calculates tunneling exponents according to the clean picture (4.4) (see Fig. 3) or the dirty scenario (3.11); we also show that there is some extra structure for negative m which is the source of the differing exponents as obtained from (3.11) versus (4.4) at $m < 0$.

1. Example for positive m : $\nu = 2/5$

First we give a simple example for $\nu = 2/5$ ($m = 2$, $p = 1$). In both $\tilde{\varphi}$ and Φ -theory the tunneling exponents for this filling are given by $S_{\text{out}} = 3$, $S_{\text{in}} = 3/5$ (see Fig. 5).

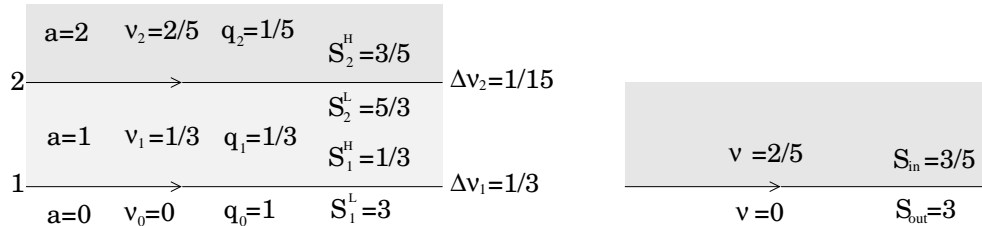


FIG. 5. Clean and dirty scenario for the $\nu=2/5$ edge. Left: spatially separated counter-flowing channels. Right: result of the K -matrix theory. Listed are filling fractions, charges and tunneling exponents.

For $\nu = 2/5$ the $\tilde{\varphi}$ -action is given by (2.17), with the K -matrix as defined in (2.16). Written explicitly we have for $A_\mu=0$

$$S_K = -\frac{i}{4\pi} \sum_{ij} \begin{pmatrix} \frac{3}{5} & -\frac{2}{5} \\ -\frac{2}{5} & \frac{3}{5} \end{pmatrix}_{ij} \oint \partial_x \tilde{\varphi}_i \partial_- \tilde{\varphi}_j - \frac{1}{2} \left(\frac{1/5}{2\pi} \right)^2 \sum_{ij} \oint_{xx'} U(x-x') \partial_x \tilde{\varphi}_i(x) \partial_{x'} \tilde{\varphi}_j(x'). \quad (4.14)$$

The diagonalized form (2.23) of this action is given by

$$S_K = -\frac{i}{4\pi} \oint \left[\frac{2}{5} \partial_x \Gamma \partial_- \Gamma + \frac{1}{2} \partial_x g \partial_- g \right] - \frac{1}{2} \left(\frac{2/5}{2\pi} \right)^2 \oint_{xx'} U(x-x') \partial_x \Gamma(x) \partial_{x'} \Gamma(x') \quad (4.15)$$

where the charged and neutral mode are defined in (2.22) and in this case read

$$\begin{pmatrix} g \\ \Gamma \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} \begin{pmatrix} \tilde{\varphi}_1 \\ \tilde{\varphi}_2 \end{pmatrix} \quad ; \quad \begin{pmatrix} \tilde{\varphi}_1 \\ \tilde{\varphi}_2 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & 1 \\ -\frac{1}{2} & 1 \end{pmatrix} \begin{pmatrix} g \\ \Gamma \end{pmatrix}. \quad (4.16)$$

On the other hand, the action for the Φ -field on the separated edges is given by (4.3). If we compress all the edges together we get (again for $A_\mu=0$)

$$S_{\text{sep}} = -\frac{i}{4\pi} \oint \left[\frac{1}{3} \partial_x \Phi_1 \partial_- \Phi_1 + \frac{1}{15} \partial_x \Phi_2 \partial_- \Phi_2 \right] - \frac{1}{2} \left(\frac{1}{2\pi} \right)^2 \oint_{xx'} U(x-x') \partial_x \left(\frac{1}{3} \Phi_1 + \frac{1}{15} \Phi_2 \right) \partial_{x'} \left(\frac{1}{3} \Phi_1 + \frac{1}{15} \Phi_2 \right). \quad (4.17)$$

We define a neutral mode γ and charged mode Γ for the Φ -theory as follows

$$\begin{pmatrix} \gamma \\ \Gamma \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ \frac{5}{6} & \frac{1}{6} \end{pmatrix} \begin{pmatrix} \Phi_1 \\ \Phi_2 \end{pmatrix} \quad ; \quad \begin{pmatrix} \Phi_1 \\ \Phi_2 \end{pmatrix} = \begin{pmatrix} \frac{1}{6} & 1 \\ -\frac{5}{6} & 1 \end{pmatrix} \begin{pmatrix} \gamma \\ \Gamma \end{pmatrix}. \quad (4.18)$$

Whereas the $\tilde{\varphi}$ fields are all equivalent in the dirty edge theory, here the Φ are not, since the spatially separated edges have unequal $\Delta\nu$ values which appear in the definition of Γ . In terms of Γ and γ the Φ -action (4.17) takes the form

$$S_{\text{sep}} = -\frac{i}{4\pi} \oint \left[\frac{2}{5} \partial_x \Gamma \partial_- \Gamma + \frac{1}{18} \partial_x \gamma \partial_- \gamma \right] - \frac{1}{2} \left(\frac{2/5}{2\pi} \right)^2 \oint_{xx'} U(x-x') \partial_x \Gamma \partial_{x'} \Gamma \quad (4.19)$$

Comparing the two results (4.15) and (4.19), we see that there is complete equivalence given the relation

$$g = \frac{1}{3} \gamma. \quad (4.20)$$

Having shown that the actions are equivalent, we now want to compare tunneling operators. In order to do this, we in principle only need to rewrite the $\tilde{\varphi}$ and Φ that appear in the various operators in terms of charged and neutral modes, and then use (4.15) or (4.19). However, much insight can be gained by also expressing $\tilde{\varphi}$ and Φ in terms of each other. Combining (4.16), (4.18) and (4.20), we find

$$\begin{pmatrix} \tilde{\varphi}_1 \\ \tilde{\varphi}_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \frac{2}{3} & \frac{1}{3} \end{pmatrix} \begin{pmatrix} \Phi_1 \\ \Phi_2 \end{pmatrix} \quad ; \quad \begin{pmatrix} \Phi_1 \\ \Phi_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -2 & 3 \end{pmatrix} \begin{pmatrix} \tilde{\varphi}_1 \\ \tilde{\varphi}_2 \end{pmatrix}. \quad (4.21)$$

In the iqHe φ -theory we can choose which one of the fields φ_1 or φ_2 to put in the tunneling operator. Eq. (3.11) then gives four possible operators, which we list in the table below, giving only the “X” in the expression $\exp -iX|_{\tau_1}^{\tau_2}$.

	$\tilde{\varphi}$ -basis	Diag. basis
Vacuum	$\varphi_1 \rightarrow \tilde{\varphi}_1$	$\Gamma + \frac{1}{6}\gamma$
	$\varphi_2 \rightarrow \tilde{\varphi}_2$	$\Gamma - \frac{1}{6}\gamma$
Bulk	$\varphi_1 \rightarrow (K^{-1}\tilde{\varphi})_1 = \frac{3}{5}\tilde{\varphi}_1 - \frac{2}{5}\tilde{\varphi}_2$	$\frac{1}{5}\Gamma + \frac{1}{6}\gamma$
	$\varphi_2 \rightarrow (K^{-1}\tilde{\varphi})_2 = -\frac{2}{5}\tilde{\varphi}_1 + \frac{3}{5}\tilde{\varphi}_2$	$\frac{1}{5}\Gamma - \frac{1}{6}\gamma$

There are only two corresponding operators in the Φ -theory. An approach from the vacuum can only involve Φ_1 because the other field does not live on the outermost edge. In the same way, the bulk operator can only contain Φ_2 . From (4.4) we get

	Φ -basis	$\tilde{\varphi}$ -basis	Diag. basis
Vacuum	Φ_1	$\tilde{\varphi}_1$	$\Gamma + \frac{1}{6}\gamma$
Bulk	$\frac{1}{5}\Phi_2$	$-\frac{2}{5}\tilde{\varphi}_1 + \frac{3}{5}\tilde{\varphi}_2$	$\frac{1}{5}\Gamma - \frac{1}{6}\gamma$

In all cases the exponents obtained from the operators in these tables are $S_{\text{out}} = 3$, $S_{\text{in}} = 3/5$. There is no a priori reason why these numbers should be the same as those appearing in Fig.5. In the short distance theory the actions for the two edges are uncoupled, while we have now used the action (4.15) where they are coupled.

2. Generalization for $m > 0$

The analysis for $\nu = 2/5$ is easily generalized to all filling fractions resulting from positive m . Let us redo all the steps in the same order.

The K -matrix action for the $\tilde{\varphi}_i$ is given by (2.17). Its diagonal form is given by (2.23) with $\text{sgn}(m) = +1$ and $\Gamma, g_1, \dots, g_{m-1}$ as defined in (2.22). For the Φ -theory we define the charged and neutral modes as follows

$$\begin{aligned}
\Gamma &= \frac{1}{\nu} \sum_{a=1}^m \Delta \nu_a \Phi_a \quad ; \quad \gamma_k = \frac{1}{\nu_k} \sum_{a=1}^k \Delta \nu_a \Phi_a - \Phi_{k+1} \\
\Phi_a &= \Gamma + \sum_{k=a}^{m-1} \frac{\Delta \nu_{k+1}}{\nu_{k+1}} \gamma_k - \frac{\nu_{a+1}}{\nu_a} \gamma_{a-1}.
\end{aligned} \tag{4.22}$$

The diagonalized form of the action reads

$$S = -\frac{i}{4\pi} \oint \left[\nu \partial_x \Gamma \partial_- \Gamma + \sum_{k=1}^{m-1} \frac{k}{k+1} q_k^2 \partial_x \gamma_k \partial_- \gamma_k \right] - \frac{\nu^2}{8\pi^2} \oint_{xx'} U \partial_x \Gamma \partial_{x'} \Gamma \tag{4.23}$$

The only difference with (2.23) is the factor q_k^2 in front of the neutral modes. The actions for $\tilde{\varphi}$ and Φ are identical if we define

$$g_k = q_k \gamma_k. \tag{4.24}$$

Using (2.22), (4.22) and (4.24) we are now able to relate the Φ_i to the $\tilde{\varphi}_i$,

$$q_k \Phi_k = \sum_{a=1}^k (K_{k \times k}^{-1})_{ka} \tilde{\varphi}_a. \quad (4.25)$$

This is a remarkable result. It shows that the fields are dually related. Notice that the K -matrix appearing in (4.25) does not have the full $m \times m$ size. In order to find the k 'th Φ -field one has to use the K -matrix for a truncated system which has only k edges, i.e. $(K_{k \times k}^{-1})_{ij} = \delta_{ij} - 2pq_k$. If one wants to write (4.25) as a vector equation of the form $(q\Phi)_i = M_{ij} \tilde{\varphi}_j$, the M is *not* given by K^{-1} but by a stacking together of rows taken from matrices $K_{k \times k}^{-1}$ of increasing size.

Given below are those tunneling operators for the clean and dirty scenario that describe the same tunneling processes.

	$\tilde{\varphi}$ -theory	Φ -theory
Vacuum	$\varphi_k \rightarrow \tilde{\varphi}_k$	$\Phi_1 = \tilde{\varphi}_1$
Bulk	$\varphi_k \rightarrow (K^{-1} \tilde{\varphi})_k$	$q_m \Phi_m = (K^{-1} \tilde{\varphi})_m$

From appendix C we know that the $\tilde{\varphi}_1, \dots, \tilde{\varphi}_m$ all give the same result $S_{\text{out}} = 2p+1$. That settles it for the vacuum operators. The beautiful thing about (4.25) is that it ensures that the bulk operator $q_m \Phi_m$ takes the form of the truly dual field $(K^{-1} \tilde{\varphi})_m$ involving the full K -matrix. From (3.14) we know that all the dual fields yield the same exponent $S_{\text{in}} = 1 - 2pq_m$.

So now we have shown for $m > 0$ that the long distance limit of the short distance Φ -theory indeed reproduces the physics of the $\tilde{\varphi}$ -theory. But why does the Φ -theory at positive m also give the same exponents *before the long distance limit is taken*? The answer is again provided by (4.25). Eq. (4.5) shows that the exponent on the isolated m 'th edge is equal to $q_m^2 / \Delta \nu_m = q_m / q_{m-1}$. On the other hand, the long distance picture leads to operators $\exp -i(K^{-1} \tilde{\varphi})_i$ in terms of fields dual to $\tilde{\varphi}_i$ which have coupling matrix K instead of K^{-1} . This leads to an exponent K_{ii}^{-1} , which is exactly equal to q_m / q_{m-1} .

3. Example for negative m : $\nu = 2/3$

For $m < 0$ the relation between the clean and dirty scenario is more subtle. In order to get the flavor, let us first consider the easiest example, $\nu = 2/3$. In Fig.6 we have shown the relevant piece of Fig.3 as compared to the K -matrix picture.

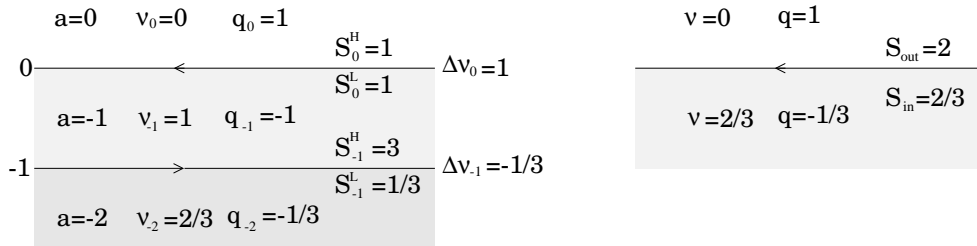


FIG. 6. Clean and dirty scenario for the $\nu = 2/3$ edge. Left: spatially separated counter-flowing channels. Right: K -matrix theory. Listed are filling fractions, charges and tunneling exponents.

Our starting point for the $\tilde{\varphi}$ -theory is again (2.17). The explicit action for $A_\mu=0$ is given by

$$S_K = -\frac{i}{4\pi} \sum_{ij} \begin{pmatrix} -\frac{1}{3} & \frac{2}{3} \\ \frac{2}{3} & -\frac{1}{3} \end{pmatrix}_{ij} \oint \partial_x \tilde{\varphi}_i \partial_- \tilde{\varphi}_j - \frac{1}{2} \left(\frac{1/3}{2\pi} \right)^2 \sum_{ij} \oint_{x,x'} U \partial_x \tilde{\varphi}_i \partial_{x'} \tilde{\varphi}_j \quad (4.26)$$

and after diagonalization it takes the form

$$-\frac{i}{4\pi} \oint \left[\frac{2}{3} \partial_x \Gamma \partial_- \Gamma - \frac{1}{2} \partial_x g \partial_- g \right] - \frac{1}{2} \left(\frac{2/3}{2\pi} \right)^2 \oint_{x,x'} U \partial_x \Gamma \partial_{x'} \Gamma. \quad (4.27)$$

The definition of the charged and neutral modes again follows from (2.22),

$$\begin{pmatrix} g \\ \Gamma \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} \begin{pmatrix} \tilde{\varphi}_1 \\ \tilde{\varphi}_2 \end{pmatrix} \quad ; \quad \begin{pmatrix} \tilde{\varphi}_1 \\ \tilde{\varphi}_2 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & 1 \\ -\frac{1}{2} & 1 \end{pmatrix} \begin{pmatrix} g \\ \Gamma \end{pmatrix}. \quad (4.28)$$

The action (4.3) for the Φ -theory with compressed edges is now given by

$$S_{\text{sep}} = -\frac{i}{4\pi} \oint \left[\partial_x \Phi_1 \partial_- \Phi_1 - \frac{1}{3} \partial_x \Phi_2 \partial_- \Phi_2 \right] - \frac{1/2}{(2\pi)^2} \oint_{x,x'} U \partial_x (\Phi_1 - \frac{1}{3} \Phi_2) \partial_{x'} (\Phi_1 - \frac{1}{3} \Phi_2) \quad (4.29)$$

Charged and neutral modes are defined in a way similar to (4.22), taking into account the different signs of the $\Delta\nu$.

$$\begin{pmatrix} \gamma \\ \Gamma \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ \frac{3}{2} & -\frac{1}{2} \end{pmatrix} \begin{pmatrix} \Phi_1 \\ \Phi_2 \end{pmatrix} \quad ; \quad \begin{pmatrix} \Phi_1 \\ \Phi_2 \end{pmatrix} = \begin{pmatrix} -\frac{1}{2} & 1 \\ -\frac{3}{2} & 1 \end{pmatrix} \begin{pmatrix} \gamma \\ \Gamma \end{pmatrix}. \quad (4.30)$$

This leads to the following diagonalized form,

$$S_{\text{sep}} = -\frac{i}{4\pi} \oint \left[\frac{2}{3} \partial_x \Gamma \partial_- \Gamma - \frac{1}{2} \partial_x \gamma \partial_- \gamma \right] - \frac{1}{2} \left(\frac{2/3}{2\pi} \right)^2 \oint_{x,x'} U \partial_x \Gamma \partial_{x'} \Gamma \quad (4.31)$$

We find that (4.27) and (4.31) are identical after defining $g = \gamma$. Using (4.28) and (4.30) we get

$$\begin{pmatrix} \tilde{\varphi}_1 \\ \tilde{\varphi}_2 \end{pmatrix} = \begin{pmatrix} 2 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \Phi_1 \\ \Phi_2 \end{pmatrix} \quad ; \quad \begin{pmatrix} \Phi_1 \\ \Phi_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} \tilde{\varphi}_1 \\ \tilde{\varphi}_2 \end{pmatrix}. \quad (4.32)$$

This relation does *not* satisfy (4.25) for the quantities $q_{-1}\Phi_1$ and $q_{-2}\Phi_2$. (Negative indices have to be taken because the relevant regions in Fig.6 are $a=-1$ and $a=-2$.) Instead we have

$$\begin{aligned} q_{-1}\Phi_1 &= \tilde{\varphi}_1 - 2\Gamma \\ q_{-2}\Phi_2 &= (K^{-1}\tilde{\varphi})_2 - \frac{2}{3}\Gamma. \end{aligned} \quad (4.33)$$

Eq. (4.25) produces the first term on the r.h.s., an expression that has the right magnitude of the charge but the wrong sign. The $q_{-1}\Phi_1$ has charge -1 , so the extra term -2Γ is precisely what is needed to flip the sign of the charge. In the same way, the term $-\frac{2}{3}\Gamma$ changes the charge from the $+1/3$ that $(K^{-1}\tilde{\varphi})_2$ yields to the $-1/3$ that is correct for the $q_{-2}\Phi_2$.

An interesting subtlety is that even though (4.25) does not hold for the relation between Φ and $\tilde{\varphi}$ fields, all tunneling exponents behave as if it did hold. The reason for this is the following. After a tunneling operator has been expressed in terms of charged and neutral modes, it is easy to see that the only effect of the extra term involving Γ is to change the sign of the factor standing in front of Γ . This sign change has no effect whatsoever on the calculation of the tunneling exponent. In the table below we explicitly write down all the tunneling operators.

	$\tilde{\varphi}$ -theory		Φ -theory		
	$\tilde{\varphi}$ -basis	Diag. basis	Φ -basis	$\tilde{\varphi}$ -basis	Diag. basis
Vacuum	$\varphi_1 \rightarrow \tilde{\varphi}_1$	$\Gamma + \frac{1}{2}\gamma$	Φ_1	$\tilde{\varphi}_2$	$\Gamma - \frac{1}{2}\gamma$
	$\varphi_2 \rightarrow \tilde{\varphi}_2$	$\Gamma - \frac{1}{2}\gamma$			
Bulk	$\varphi_1 \rightarrow \frac{1}{3}\tilde{\varphi}_1 - \frac{2}{3}\tilde{\varphi}_2$	$-\frac{1}{3}\Gamma + \frac{1}{2}\gamma$	$-\frac{1}{3}\Phi_2$	$\frac{1}{3}\tilde{\varphi}_1 - \frac{2}{3}\tilde{\varphi}_2$	$-\frac{1}{3}\Gamma + \frac{1}{2}\gamma$
	$\varphi_2 \rightarrow -\frac{2}{3}\tilde{\varphi}_1 + \frac{1}{3}\tilde{\varphi}_2$	$-\frac{1}{3}\Gamma - \frac{1}{2}\gamma$			

In both the Φ -theory and $\tilde{\varphi}$ -theory the tunneling exponents obtained from the above expressions are

$$S_{\text{out}} = 2 \quad ; \quad S_{\text{in}} = 2/3. \quad (4.34)$$

The discrepancy with the short distance clean scenario result $S_{\text{out}} = 1$, $S_{\text{in}} = 1/3$ is caused by the fact that (4.32) does not satisfy (4.25), so that the argument given in the positive m subsection does not apply.

4. Generalization for arbitrary sign of m

For completeness, we now present general equations that comprise all the above cases. The $\tilde{\varphi}$ -theory expressions (2.23) and (2.22) are equally valid for negative and positive m . No extra work is needed there. The Φ -theory expressions (4.22) and (4.23) do need some modification if we want to incorporate the possibility of m being negative. Eq. (4.22) for the charged and neutral modes can be generalized as follows,

$$\begin{aligned} \Gamma &= \frac{1}{\nu} \sum_{a=1}^{|m|} \Delta\nu_{[a]} \Phi_a \quad ; \quad \gamma_k = \frac{1}{\nu_{[k]}} \sum_{a=1}^k \Delta\nu_{[a]} \Phi_a - \Phi_{k+1} \\ \Phi_a &= \Gamma + \sum_{k=a}^{|m|-1} \frac{\Delta\nu_{[k+1]}}{\nu_{[k+1]}} \gamma_k - \frac{\nu_{[a+1]}}{\nu_{[a]}} \gamma_{a-1} \end{aligned} \quad (4.35)$$

Here the notation $\nu_{[a]}$ (with $a > 0$) means ν_a for samples with positive m and ν_{-a} for $m < 0$ samples. For instance, in case of negative m we would have $\Delta\nu_{[2]} = \nu_{-2} - \nu_{-1}$.

Using this definition, the generalization of (4.23) becomes

$$S = -\frac{i}{4\pi} \oint \left[\nu \partial_x \Gamma \partial_- \Gamma + \text{sgn}(m) \sum_{k=1}^{|m|-1} \frac{k}{k+1} q_{[k]}^2 \partial_x \gamma_k \partial_- \gamma_k \right] - \frac{\nu^2}{8\pi^2} \oint_{xx'} U \partial_x \Gamma \partial_{x'} \Gamma \quad (4.36)$$

Comparing this expression to (2.23) we can make the identification

$$g_k = |q_{[k]}| \gamma_k. \quad (4.37)$$

This enables us to relate the fields $\tilde{\varphi}_i$ to the Φ_a .

$$\tilde{\varphi}_k = \text{sgn}(m) \left[q_{[k]} \Phi_k + \sum_{a=1}^k 2p \Delta \nu_{[a]} \Phi_a \right] + [1 - \text{sgn}(m)] \cdot \Gamma \quad (4.38)$$

$$\begin{aligned} q_{[k]} \Phi_k &= \text{sgn}(m) \tilde{\varphi}_k - 2p q_{[k]} \sum_{a=1}^k \tilde{\varphi}_a + [1 - \text{sgn}(m)] \cdot q_{[k]} \Gamma \\ &= \sum_{a=1}^k (K_{k \times k}^{-1})_{ka} \tilde{\varphi}_a + [1 - \text{sgn}(m)] \cdot q_{[k]} \Gamma \end{aligned} \quad (4.39)$$

where $K_{k \times k}$ again denotes the K -matrix for a system containing k edges. It is easily checked that for positive m (4.25) is reobtained.

Eq. (4.39) shows that what happened at $\nu=2/3$ happens at all negative m states: The $K^{-1} \tilde{\varphi}$ part of the tunneling expression $q_m \Phi_{|m|}$ has the right magnitude of the charge q_m but the wrong (=positive) sign. The extra term $2q_m \Gamma$ flips the sign. The reason why we write Γ in (4.39) instead of expressing it in terms of the $\tilde{\varphi}_a$ is that we can now directly discern the effect on the tunneling operators. In the diagonal basis the effect of the extra Γ term is to flip the sign of the coefficient of Γ in the tunneling operators. Just as in the $2/3$ case this leaves the exponent unaffected.

Also generalized to all $m < 0$, the discrepancy between the exponents as calculated in the short and long distance limits is due to the fact that (4.25) does not hold,

D. Quasiparticles and the Laughlin argument

The particles probed in tunneling experiments are not necessarily the same as those featuring in Laughlin's flux argument.

In the integer case, the effect of putting a flux quantum through a hole in the sample is a transport of one electron through each Landau level below the Fermi energy. In the language of (4.1), considering for simplicity $m > 0$, what would happen at the a 'th edge is that $a-1$ electrons enter that edge from below and that a electrons leave it on the upper side. In this way, a total of m electrons gets transported into the bulk, which is the correct number. The operator describing this process is given by

$$\exp -i \int_{\tau_1}^{\tau_2} d\tau \left\{ a [\partial_0 \varphi - a_0](\vec{x}_H) - [a-1] [\partial_0 \varphi - a_0](\vec{x}_L) \right\} \quad (4.40)$$

where \vec{x}_H lies infinitesimally above the a 'th edge and \vec{x}_L lies below it. Both points have the same x -coordinate.

Knowing the results of section IV A, it is easy to do the Chern-Simons mapping. The constraint equation multiplied by a_- contains the sum of two delta functions instead of just one. Correspondingly, the redefined field Φ contains the sum of two arctg functions. The mapped expression is given by

$$\exp -i \int_{\tau_1}^{\tau_2} d\tau \left\{ \nu_a \partial_0 \Phi(\vec{x}_H) - \nu_{a-1} \partial_0 \Phi(\vec{x}_L) \right\}. \quad (4.41)$$

As is clearly seen from the constants appearing here, the emitted charge is ν_a and the absorbed charge is ν_{a-1} . Taking all the edges together we have a total transported charge of $\sum_{a=1}^m \Delta\nu_a = \nu_m$, which is the correct number for a system of filling fraction ν_m . Taking the limit where \vec{x}_H and \vec{x}_L go to the same point x on the edge, the tunneling expression takes the simple form

$$\exp -i \int_{\tau_1}^{\tau_2} d\tau \Delta\nu_a \partial_0 \Phi(\tau, x). \quad (4.42)$$

V. EDGE TUNNELING AT ARBITRARY ν

In [III] we discussed how the non-quantized tunneling exponent⁵ $1/\nu$ can be understood as a consequence of the Coulomb interactions between the edge modes and the localized states present in the bulk. For ν around integer filling, we explicitly showed how the interactions render the neutral modes irrelevant. Here we will generalize this result to fractional fillings, in particular to Jain's main hierarchy.

At almost integer filling $\nu = m + \varepsilon$, a fraction ε of the total area is occupied by regions with filling $m+1$, while the rest of the sample has filling m . After performing the flux attachment, a fraction $1 - \varepsilon$ of the area has filling ν_m and a fraction ε has ν_{m+1} , leading to

$$\nu = m + \varepsilon \xrightarrow{\text{CS mapping}} \nu = \nu_m + \varepsilon(\nu_{m+1} - \nu_m). \quad (5.1)$$

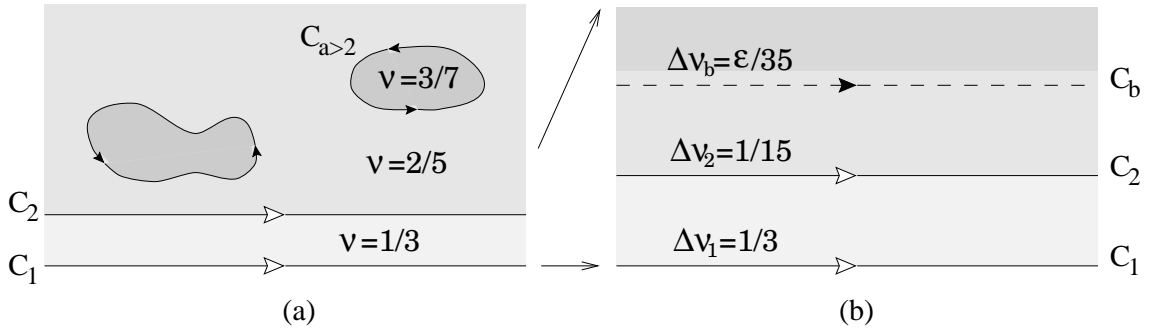


FIG. 7. Example of the long wavelength limit for filling fraction $\nu = 2/5 + \varepsilon/35$. (a) The $3/7$ islands occupy a fraction ε of the total area. (b) Long wavelength limit. The bulk degrees of freedom are effectively represented by an extra chiral boson on the contour C_b with charge density $\frac{\varepsilon}{35} \partial_x \Phi|_{C_b}$. The filling fraction is expressed as $\nu = \Delta\nu_1 + \Delta\nu_2 + \Delta\nu_b$.

The action for this situation is given by (4.3), but now including the bulk contours $C_{a>m}$. In [III] we showed that in the long wavelength limit the bulk degrees of freedom can effectively be described by an extra chiral boson on a contour C_b near the edge plus an edge term

$$S_n = -\alpha \int_{C_b} dk |k| \partial_\perp \Phi(-k) \partial_\perp \Phi(k) \quad (5.2)$$

(with ∂_\perp the derivative normal to C_b and α a positive constant), which suppresses the neutral modes. The only difference between the integer and fractional case lies in the edge parameters $\Delta\nu_1, \dots, \Delta\nu_m$ which were all equal to 1 in the integer case and $\Delta\nu_b := \varepsilon(\nu_{m+1} - \nu_m)$ which was equal to ε . In terms of fields Φ_1, \dots, Φ_m and $\Phi_{m+1} := \Phi_b$ which live on *one* edge, the long wavelength limit yields the following action

$$\begin{aligned} S[\Phi] = & -\frac{i}{4\pi} \sum_{a=1}^{m+1} \delta_a \oint \partial_x \Phi_a \partial_- \Phi_a - \frac{1}{8\pi^2} \sum_{a,b=1}^{m+1} \delta_a \delta_b \oint_{xx'} U(x-x') \partial_x \Phi_a \partial_{x'} \Phi_b \\ & - \frac{1}{8\pi^2} \sum_{a,b=1}^{m+1} V_{ab} \delta_a \delta_b \oint \partial_x \Phi_a \partial_x \Phi_b + S_n. \end{aligned} \quad (5.3)$$

Here we have defined $\delta_a = \Delta\nu_a$ for $a = 1, \dots, m$ and $\delta_{m+1} = \varepsilon \Delta\nu_{m+1}$. All short range effects, originating from the fact that the bulk orbitals sometimes venture near the edge, are contained in the matrix V .

The definition of a charged mode Γ and neutral modes $\gamma_1, \dots, \gamma_m$ is similar to (4.22),

$$\Gamma = \frac{1}{\nu} \sum_{a=1}^{m+1} \delta_a \Phi_a \quad ; \quad \gamma_k = \frac{1}{\nu_k} \sum_{a=1}^k \delta_a \Phi_a - \varphi_{k+1}. \quad (5.4)$$

The Φ 's are expressed in terms of the γ 's as follows

$$\Phi_a = \Gamma - \frac{\nu_{a-1}}{\nu_a} \gamma_{a-1} + \sum_{k=a}^m \frac{\delta_{k+1}}{\nu_{k+1}} \gamma_k. \quad (5.5)$$

Written in the new basis, the action (5.3) is given by

$$\begin{aligned} S[\Gamma, \gamma] = & S_n[\gamma] - \frac{i}{4\pi} \oint \left[\nu \partial_x \Gamma \partial_- \Gamma + \sum_{k=1}^m \delta_{k+1} \frac{\nu_k}{\nu_{k+1}} \partial_x \gamma_k \partial_0 \gamma_k \right] \\ & - \frac{\nu^2}{8\pi^2} \oint_{xx'} U(x, x') \partial_x \Gamma \partial_{x'} \Gamma - \frac{1}{8\pi^2} \sum_{a,b=1}^m \oint V_{ab} \partial_x \gamma_a \partial_x \gamma_b. \end{aligned} \quad (5.6)$$

The neutral modes do not contribute to a calculation of the tunneling density of states because of (5.2) and consequently the parameter ν in front of the $\partial_x \Gamma \partial_- \Gamma$ term determines the tunneling exponent. The result is $S = 1/\nu$, the same result that we obtained for the integer theory.

VI. PLATEAU TRANSITIONS

We have discussed integer plateau transitions in [III]. Recapitulating the results obtained there, we can sketch a transition from $\nu=m$ to $\nu=m+1$ (with m positive) as follows. One has m copies of the $\nu=1$ type Q -edge theory on the edge of the sample and one copy percolating through the bulk. Upon taking the continuum limit, the percolating “edge” gives rise to a nonlinear sigma model with $\sigma_{xx} = \frac{1}{2}$ and $\sigma_{xy} = \frac{1}{2}$. Combining the edge and bulk contributions, one gets a plateau transition at $\sigma_{xx} = \frac{1}{2}$, $\sigma_{xy} = m + \frac{1}{2}$.

We now generalize this picture to the fractional case, in particular to the transition $\nu_n \rightarrow \nu_{n+1}$. At the boundary of the sample we have n edge channels with total contribution $(\nu_1 - \nu_0) + (\nu_2 - \nu_1) + \dots + (\nu_n - \nu_{n-1}) = \nu_n$ to the Hall conductance. One edge channel with parameter $\Delta\nu = \nu_{n+1} - \nu_n$ (see Fig.7a) percolates through the bulk. In order to find out what kind of sigma-model arises from this percolating “edge”, we first we go back to the Q -field formalism from (4.3). As mentioned in section II A, we obtained a theory for chiral bosons starting from an action for the matrix field Q coupled to the external field A_μ . For completeness, we give the the action $S_{\Delta\nu}[Q, A]$ for one edge channel with parameter $\Delta\nu$,

$$\begin{aligned} S_{\Delta\nu}[Q, A] = & S_{\text{top}}[Q] + \frac{i}{4\pi} \left[\sum_{\alpha} \int \nu (A^{\text{eff}})^{\alpha} \wedge d(A^{\text{eff}})^{\alpha} - \Delta\nu \oint dx A_x^{\dagger} A_c^{\text{eff}} \right] \\ & + \frac{\pi}{4\beta v_d} S_F[Q] - \frac{\pi}{4\beta} \sum_{n\alpha} \int dk_x \frac{1}{\tilde{v}(k_x)} \left| \text{tr} I_{-n}^{\alpha} Q - \frac{\beta}{\pi} \sqrt{\Delta\nu} (A_c^{\text{eff}})^{\alpha}_n \right|^2 \\ & + \frac{\beta}{2} \left(\frac{1}{2\pi} \right)^2 \int d^2x d^2x' \nu(\vec{x}) B(\vec{x}) U(\vec{x} - \vec{x}') \nu(\vec{x}') B(\vec{x}') \end{aligned} \quad (6.1)$$

where in the definition of A_0^{eff} (2.5) the m now has to be replaced by $\nu(\vec{x})$, and in the definition of \tilde{v} (2.26) one should read $\Delta\nu$ instead of ν . We have neglected here the Coulomb interaction with other channels.

Notice that in (6.1) the charge of the Q -field is no longer 1 but $\sqrt{\Delta\nu}$. A gauge transformation takes the form

$$A_\mu \rightarrow A_\mu + \partial_\mu \chi \quad ; \quad Q \rightarrow e^{i\sqrt{\Delta\nu}\cdot\hat{\chi}} Q e^{-i\sqrt{\Delta\nu}\cdot\hat{\chi}}. \quad (6.2)$$

Following [III] we use (6.1) and obtain the σ model action for the percolating network. Putting $A_\mu=0$, the result for the kinetic terms is equal to

$$S_{\text{cont}}[Q] = -\frac{1}{8} \tilde{\sigma}_{xx} \text{Tr} (\nabla Q)^2 + \frac{1}{8} \tilde{\sigma}_{xy} \varepsilon^{ij} \text{Tr} Q \partial_i Q \partial_j Q \quad (6.3)$$

where

$$\tilde{\sigma}_{xx} = \frac{1}{2} \quad ; \quad \tilde{\sigma}_{xy} = \frac{1}{2} \quad (6.4)$$

now represent the mean field conductances of the *composite fermions*. The *actual* mean field values obtained from linear response in the external fields are different by a factor of $\Delta\nu$, i.e.

$$\sigma'_{xx} = \frac{1}{2} \Delta\nu \quad ; \quad \sigma_{xy}^{\text{bulk}} = \frac{1}{2} \Delta\nu \quad (6.5)$$

Adding the edge contribution to this result, we finally get for the mean field conductances at the plateau transition

$$\sigma'_{xx} = \frac{1}{2} \cdot \frac{1}{(2pn+p+1)^2-p^2} \quad ; \quad \sigma'_{xy} = \frac{1}{2}(\nu_n + \nu_{n+1}) = \frac{(2pn+p+1)(n+1/2)-p/2}{(2pn+p+1)^2-p^2}. \quad (6.6)$$

This result differs from what one would expect from acting with $\text{Sl}(2, \mathbb{Z})$ on an integer plateau transition. In general, the $\text{Sl}(2, \mathbb{Z})$ mapping of the conductances takes the form

$$\sigma \rightarrow \sigma' = \frac{\sigma}{2p\sigma+1}, \quad (6.7)$$

where the complex number σ is defined as $\sigma = \sigma_{xy} + i\sigma_{xx}$. In terms of σ_{xy} and σ_{xx} (6.7) reads

$$\sigma'_{xx} = \frac{\sigma_{xx}}{(2p\sigma_{xy}+1)^2+(2p\sigma_{xx})^2} \quad ; \quad \sigma'_{xy} = \frac{\sigma_{xy}(2p\sigma_{xy}+1)+2p\sigma_{xx}^2}{(2p\sigma_{xy}+1)^2+(2p\sigma_{xx})^2} \quad (6.8)$$

Inserting the composite fermion values $\sigma_{xx} = \frac{1}{2}$, $\sigma_{xy} = n + \frac{1}{2}$ we now obtain

$$\sigma'_{xx} = \frac{1}{2} \cdot \frac{1}{(2pn+p+1)^2+p^2} \quad ; \quad \sigma'_{xy} = \frac{(2pn+p+1)(n+1/2)+p/2}{(2pn+p+1)^2+p^2}. \quad (6.9)$$

The different results (6.6) and (6.9) indicate that the Chern-Simons mapping is not uniquely described by $\text{Sl}(2, \mathbb{Z})$ but depends on the details of disorder.

VII. SUMMARY AND CONCLUSIONS

We have derived, from first principles, the complete Luttinger liquid theory for abelian quantum Hall states, taking into account the presence of gauge fields and the effects of disorder and Coulomb interactions. Building on the results obtained in [I] and [III], we have applied the Chern-Simons flux attachment procedure to the theory of chiral edge bosons in the integral quantum Hall regime. The resulting action has the well known K -matrix structure, but possesses new properties and offers many novel insights due to the microscopic nature of its derivation.

We have shown that the Chern-Simons procedure is ill-defined for systems with counter-flowing edge modes, unless there are long-ranged electron-electron interactions present to stabilize the action for the charged mode. Without long-ranged interactions, the Hamiltonian of the charged mode is unbounded from below. Our description for these systems is not plagued by non-universalities (a problem that phenomenological theories have), because our velocity matrix is diagonal simultaneously with K . We also find no ambiguities concerning the definition of the Hall conductance on the edge. The chiral anomaly provides an elegant way of using Laughlin's gauge argument.

Our treatment of tunneling operators has shown that the CS procedure affects the charge and statistics of tunneling particles. An electron operator outside the sample retains its unit charge after the mapping, but inside the charge is mapped to ν/m . The operators inside and outside are related by a T-duality transformation that inverts the compactification radius (K -matrix); thus we have found a new geometrical interpretation for this duality.

Our analysis of spatially separated edge modes has revealed an even richer duality structure in the spectrum of quasiparticles, where the reversal inside/outside is generalized to a

reversal of the order of edge channels. Samples with positive m and negative m are jointly described in this picture.

We have shown that in the limit of large length scales, the theory of “clean”, spatially separated edges describes identical physics as the “dirty” K -matrix theory. In the case of counter-flowing modes this happens in a quite nontrivial way.

Following [III], we have derived an effective Luttinger liquid theory for tunneling processes into the edge at filling fractions which do not lie at plateau centers. The long range Coulomb interactions between edge and bulk states result in a suppression of the neutral modes, which directly yields a continuous tunneling exponent $1/\nu$, in accordance with experiments.

We have made a short analysis of transitions between fqH plateaus, employing the Q -field theory for percolating fractional edges. The results indicate that the critical aspects of the plateau transitions are quite generally the same for both the integral and fractional regime. However, the Chern-Simons mapping is not uniquely described by $Sl(2, \mathbb{Z})$.

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Appendix A: Integrating out the Chern-Simons field

The remnant of the noninteracting part of the action (2.11) after a_- has been integrated out is given by

$$\begin{aligned} -4\pi i \cdot S_{\text{remnant}}^{\text{free}} = s_m \sum_i \int_{\infty} [2D_- \varphi_i \nabla \times (\theta \vec{D} \varphi_i) - \theta \vec{D} \varphi_i \times \partial_- \vec{D} \varphi_i] \\ - \frac{1}{2p} \int_{\infty} (2pm\theta + 1) \vec{a} \times \partial_- \vec{a} - 2m \int_{\infty} \theta \vec{a} \times \vec{E} \end{aligned} \quad (\text{A1})$$

where θ is shorthand for $\theta(y)$ and s_m stands for $\text{sgn}(m)$. We now have to substitute (2.13) for \vec{a} ,

$$\vec{a} = \frac{2p}{2pm\theta+1} s_m [\theta \sum_i \vec{D} \varphi_i + \nabla \Omega]. \quad (\text{A2})$$

Let us first concentrate on all the terms that do not depend on the arbitrary gauge Ω . From the $\vec{a} \times \partial_- \vec{a}$ in (A1) we obtain a term quadratic in $D_{\mu} \varphi$, and the last term in (A1) produces a product of A_{μ} and $\vec{D} \varphi$. Together this is

$$-2p \sum_{ij} \int_{\infty} \frac{\theta^2}{2pm\theta+1} \vec{D} \varphi_i \times \partial_- \vec{D} \varphi_j - 4p|m| \int_{\infty} \frac{\theta^2}{2pm\theta+1} \sum_i \vec{D} \varphi_i \times \vec{E}. \quad (\text{A3})$$

The expression involving step functions can be simplified to $\theta\nu/m$, since $\partial_y[\theta^2/(2pm\theta+1)] = \partial_y[\theta\nu/m]$. Adding (A3) to the first line of (A1), we obtain

$$-4\pi i \cdot S_{\Omega=0}^{\text{free}} = \nu \int A \wedge dA - \sum_{ij} [\delta_{ij} s_m - 2p \frac{\nu}{m}] \oint (D_x \varphi_i D_- \varphi_j - \varphi_i E_x). \quad (\text{A4})$$

Now we concentrate on the Ω . From $\vec{a} \times \partial_- \vec{a}$ we get a quadratic term in Ω ,

$$-2p \int_{\infty} \frac{1}{2pm\theta+1} \nabla \Omega \times \partial_- \nabla \Omega. \quad (\text{A5})$$

Integrating by parts and using the relation $\partial_y[1/(2pm\theta+1)] = -2p\nu\delta(y)$ this can be rewritten as

$$-(2p)^2\nu \oint \partial_x \Omega \partial_- \Omega. \quad (\text{A6})$$

Both terms in the second line of (A1) give rise to linear terms in Ω ; written together this is

$$-4p s_m \int_{\infty} \frac{\theta}{2pm\theta+1} \nabla \Omega \times \nabla \sum_i D_- \varphi_i = 4p \frac{\nu}{|m|} \oint \partial_x \Omega \sum_i D_- \varphi_i, \quad (\text{A7})$$

where we have integrated by parts and used $\partial_y[\theta/(2pm\theta+1)] = \delta(y)\nu/m$. The free action after the CS mapping is given by (A4)+(A6)+(A7). It is easily seen that the terms involving Ω can be completely absorbed into a redefinition of the φ fields according to

$$\tilde{\varphi}_i = \varphi_i - 2p s_m \Omega. \quad (\text{A8})$$

The mapping of the interaction term is simple. We write the Coulomb contribution in the form

$$-\frac{1}{8\pi^2} \int_{\infty} U(\vec{x} - \vec{x}') \nabla \times [\theta \sum_i \vec{D} \varphi_i](\vec{x}) \nabla' \times [\theta \sum_j \vec{D} \varphi_j](\vec{x}') \quad (\text{A9})$$

and note that in this expression $\theta \sum_i [\vec{D} \varphi_i - \vec{a}]$, after substituting (A2), is equivalent to $\theta \frac{\nu}{m} \sum_i [\vec{D} \varphi_i - 2p s_m \Omega] = \theta \frac{\nu}{m} \sum_i \vec{D} \tilde{\varphi}_i$. With this last ingredient, the mapping is complete and yields the result (2.17).

Appendix B: Integrating out the CS field in the presence of a tunneling term

Integration over the CS gauge fields in the presence of a tunneling term proceeds along the same lines as in Appendix A. The remnant of the free action is now given by

$$\begin{aligned} -4\pi i \cdot S_{\text{remnant}}^{\text{free}} &= s_m \sum_i \int_{\infty} \left[2D_- \varphi_i \nabla \times (\theta \vec{D} \varphi_i) - \theta \vec{D} \varphi_i \times \partial_- \vec{D} \varphi_i \right] \\ &- \frac{1}{2p} \int_{\infty} (2pm\theta + 1) \vec{a} \times \partial_- \vec{a} - 2m \int_{\infty} \theta \vec{a} \times \vec{E} - 4\pi \int_{\tau_1}^{\tau_2} d\tau (\partial_0 \varphi_i - i v_d a_x)(\tau, \vec{x}_0) \end{aligned} \quad (\text{B1})$$

and we have to substitute

$$\vec{a}(\vec{x}) = \frac{2p s_m}{2pm\theta(y)+1} \left[\theta(y) \sum_i \vec{D} \varphi_i + \nabla \Omega + s_m L \nabla \arctan \frac{x-x_0}{y-y_0} \right] \quad (\text{B2})$$

with L_n as defined in (3.7). Eq. (B2) would suggest that one can obtain an action of precisely the same form as in Appendix A from (B1) (minus the tunneling term at \vec{x}_0) by

redefining Ω according to $\Omega' = \Omega + s_m \text{Larctg} \frac{x-x_0}{y-y_0}$. However, whereas Ω satisfies $\nabla \times \nabla \Omega = 0$, we have $\nabla \times \nabla \text{arctg} \frac{x-x_0}{y-y_0} = -2\pi \delta(\vec{x} - \vec{x}_0)$. This means that apart from the free part of (2.17) (containing Ω' instead of Ω) we get extra contributions at \vec{x}_0 . The $\vec{a} \times \partial_- \vec{a}$ part of (B1) gives rise to the following

$$- 2p s_m \int_{\infty} \frac{1}{2pm\theta+1} L \nabla \text{arctg} \frac{x-x_0}{y-y_0} \times \partial_- \left[s_m L \nabla \text{arctg} \frac{x-x_0}{y-y_0} + 2\theta \sum_i \nabla \varphi_i + 2\nabla \Omega \right]. \quad (\text{B3})$$

Integrating by parts and discarding all derivatives of step functions (which yield δ terms that we have already accounted for), we move all the derivatives in $[\dots]$ to the left so that they can operate on the arctg. This gives

$$4\pi p \int_{\tau_1}^{\tau_2} d\tau \left[2\frac{\nu}{m} \theta \sum_i \partial_- \varphi_i + 2s_m(1 - 2p\nu\theta) \partial_- \{L \text{arctg} \frac{x-x_0}{y-y_0}\} + 2(1 - 2p\nu\theta) \partial_- \Omega \right] (\vec{x}_0). \quad (\text{B4})$$

Notice the factor of 2 that has popped up in front of the arctg term. Its origin lies in the fact that ∂_x and ∂_y do not commute when acting on arctg. We have

$$\partial_x \partial_y = \nabla \times \nabla + \partial_y \partial_x \quad (\text{B5})$$

and the $\nabla \times \nabla$ produces an extra delta function. From the $\vec{a} \times \vec{E}$ part of (B1) we get

$$2m \int_{\infty} \theta A_- \nabla \times \vec{a} \longrightarrow -8\pi p\nu\theta \int_{\tau_1}^{\tau_2} d\tau A_- (\vec{x}_0). \quad (\text{B6})$$

The $\int d\tau$ part of (B1) gives

$$- 4\pi \int_{\tau_1}^{\tau_2} d\tau \left\{ \partial_0 \varphi_i - i v_d 2p s_m \left[\frac{\nu}{m} \theta \sum_j D_x \varphi_j + (1 - 2p\nu\theta) \partial_x (\Omega + s_m \text{Larctg} \frac{x-x_0}{y-y_0}) \right] \right\} \quad (\text{B7})$$

Adding (B4), (B6) and (B7) we find

$$S_{\text{at } \vec{x}_0}^{\text{free}} = -i \int_{\tau_1}^{\tau_2} d\tau \left\{ \partial_0 \tilde{\varphi}_i(\vec{x}_0) - 2p\frac{\nu}{m} \theta(y_0) \sum_j D_0 \tilde{\varphi}_j(\vec{x}_0) \right\} \quad (\text{B8})$$

where we have defined

$$\tilde{\varphi}_i(\vec{x}) = \varphi_i(\vec{x}) - 2ps_m \Omega(\vec{x}) - 2p \text{Larctg} \frac{x-x_0}{y-y_0}. \quad (\text{B9})$$

Now we deal with the interaction term, in the same way as in Appendix A. Again we can write

$$\theta \sum_i (\vec{D} \varphi_i - \vec{a}) \longrightarrow \theta \frac{\nu}{m} \sum_i \vec{D} \tilde{\varphi}_i. \quad (\text{B10})$$

Only one subtlety is left: because of the presence of the arctg in $\tilde{\varphi}$, the relation $\nabla \times \vec{D} \tilde{\varphi}_i = -B$ gets altered. We have

$$\nabla \times \vec{D}\tilde{\varphi}_i = -B' \quad ; \quad B'(\vec{x}) = \nabla \times \vec{A}(\vec{x}) - 2p\delta(\vec{x} - \vec{x}_0) \quad (\text{B11})$$

which reflects the fact that there is extra flux at \vec{x}_0 .

Appendix C: Tunneling exponent resulting from neutral modes

The result S_{out} (3.12) is obtained from the expectation value $\langle \exp -i\tilde{\varphi}_k |_{\tau_1}^{\tau_2} \rangle$ as follows. The field $\tilde{\varphi}_k$ decomposes into charged and neutral modes according to (2.22). From the action (2.23) we find the correlations for the diagonal modes,

$$\langle \Gamma(\tau_2)\Gamma(\tau_1) \rangle = \frac{1}{\nu} \ln(\tau_2 - \tau_1) \quad ; \quad \langle \gamma_k(\tau_2)\gamma_{k'}(\tau_1) \rangle = \delta_{kk'} \frac{k+1}{k} \ln(\tau_2 - \tau_1) \quad (\text{C1})$$

and $\langle \gamma_k \Gamma \rangle = 0$.

Evaluation of $\langle \exp -i\tilde{\varphi}_k |_{\tau_1}^{\tau_2} \rangle$ for arbitrary k then yields the tunneling exponent

$$\begin{aligned} S_{\text{out}} &= \frac{1}{\nu} + \frac{k-1}{k} + \sum_{a=k}^{|m|-1} \frac{1}{a(a+1)} = \frac{1}{\nu} + 1 - \frac{1}{k} + \sum_{a=k}^{|m|-1} \left(\frac{1}{a} - \frac{1}{a+1} \right) \\ &= 2p + 1 + \frac{1}{m} - \frac{1}{|m|}. \end{aligned} \quad (\text{C2})$$

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